ALMA : Fourier phase analysis made possible

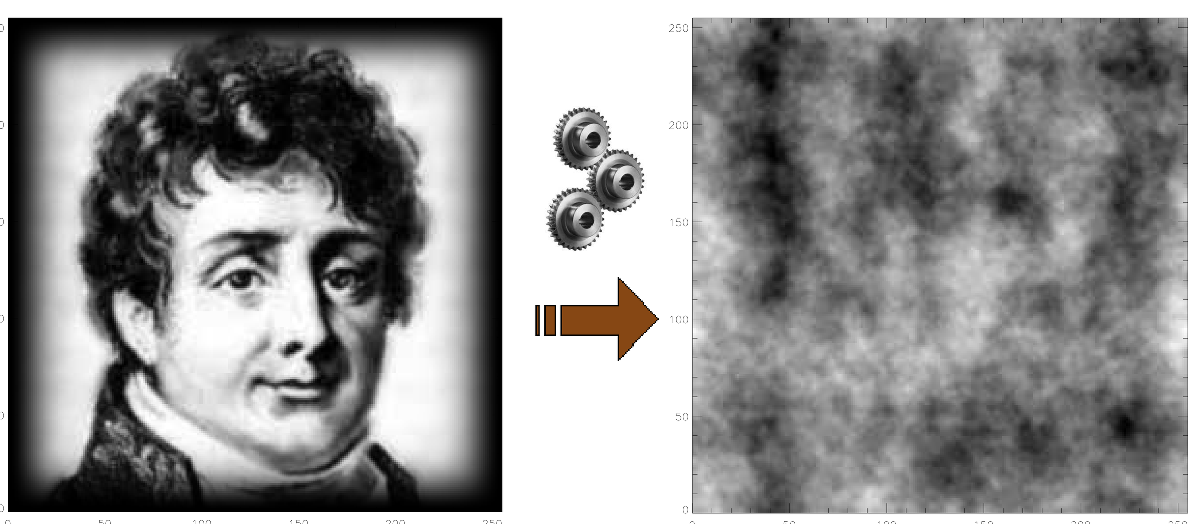
François Levrier (LERMA / ENS Paris)
Edith Falgarone (LERMA / ENS Paris)
François Viallefond (LERMA / Observatoire de Paris)

l'Observatoire de Paris — LERMA — Laboratoire d'Étude du Rayonnement et de la Matière en Astrophysique

ABSTRACT

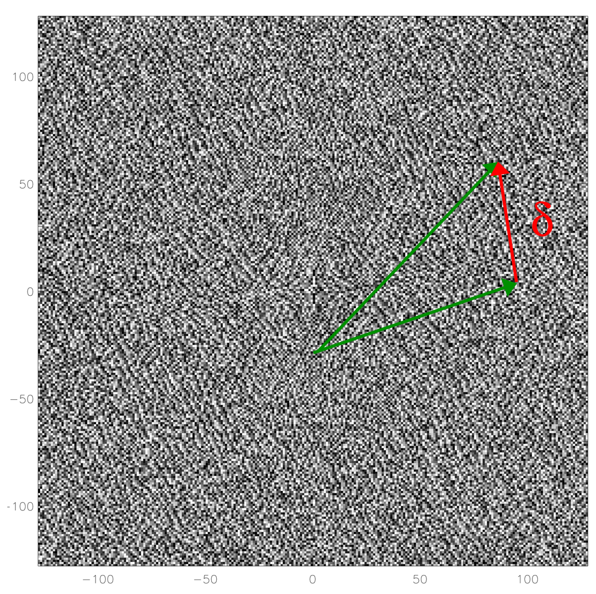
Fourier phases contain a vast amount of information about structure in direct space, that most statistical tools never tap into. We address ALMA's ability to detect and recover this information, using the probability distribution function of phase increments and the related concepts of phase entropy and phase structure quantity. We show that ALMA will be able to achieve significant detection of phase structure, and that it will do so even in the presence of a fair amount of atmospheric phase fluctuations. We also show that ALMA should be able to recover the actual "amount" of phase structure in the noise-free case, if multiple configurations are used.

1 Why study the statistics of Fourier phases?



Take an image such as the portrait of Joseph Fourier on the left. Fourier-transform it, keep the map of amplitudes, and reshuffle the map of phases randomly. Fourier-transform back to direct space and you get an image such as the one on the right. The original structure is completely lost in the process. Yet both images share the **same power spectrum**. They also share the same - essentially uniform - probability distribution function (PDF) of the phases. It is in the **Fourier-spatial distibution of the phases**, in their correlations, that information about structures in direct space should be sought.

2 Phase increments, phase entropy, phase structure quantity and von Mises distributions



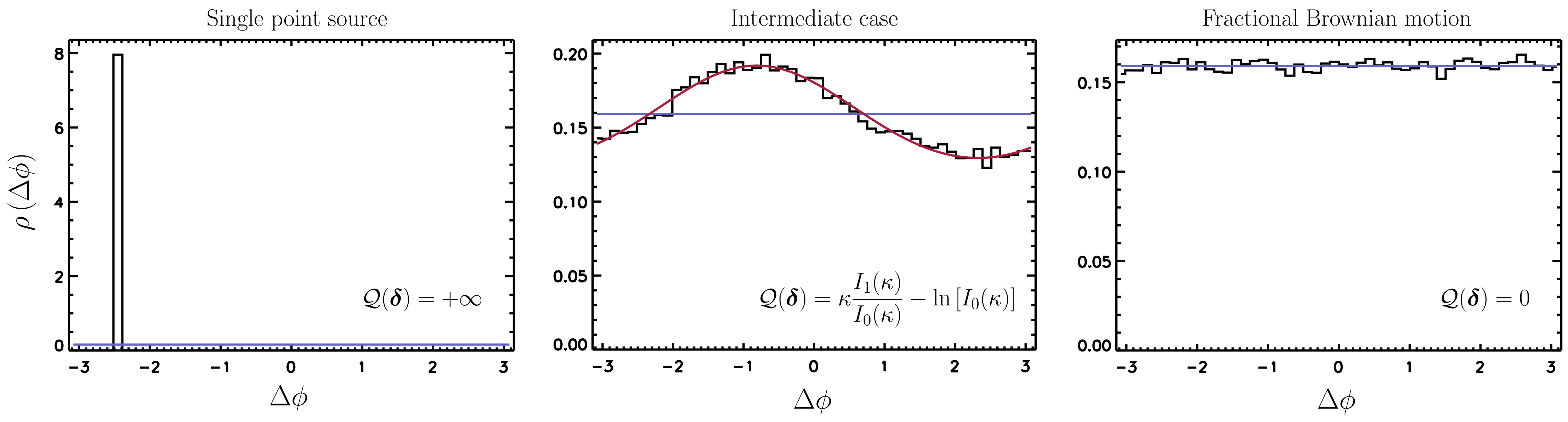
- Phase increments are defined by $\boxed{\Delta\phi = \phi(\boldsymbol{k}+\boldsymbol{\delta}) - \phi(\boldsymbol{k})}$ for a given lag vector $\boldsymbol{\delta}$.
- The probability distribution function of phase increments $\rho(\Delta\phi)$ is in general not uniform. Its shape is that of a von Mises distribution:

$$\rho(\Delta\phi) = \frac{1}{2\pi I_0(\kappa)} \exp\left[-\kappa \cos(\Delta\phi - \mu)\right]$$

- The non-uniformity of the PDF of phase increments may be quantified by

Phase entropy $\boxed{\mathcal{S}(\boldsymbol{\delta}) = -\int_{-\pi}^{\pi} \rho(\Delta\phi) \ln\left[\rho(\Delta\phi)\right] \mathrm{d}\Delta\phi}$

Phase structure quantity $\boxed{\mathcal{Q}(\boldsymbol{\delta}) = \ln(2\pi) - \mathcal{S}(\boldsymbol{\delta}) \geqslant 0}$



Typical values

$\mathcal{Q} \sim 10^{-2}$ for column densities of compressible turbulence simulations (Porter *et al.*, 1994)
$\mathcal{Q} \sim 10^{-1}$ for gravitational clustering simulations (Chiang & Coles, 2000)

3 Phase structure quantity and statistical noise

The trouble with not having infinitely many antennae

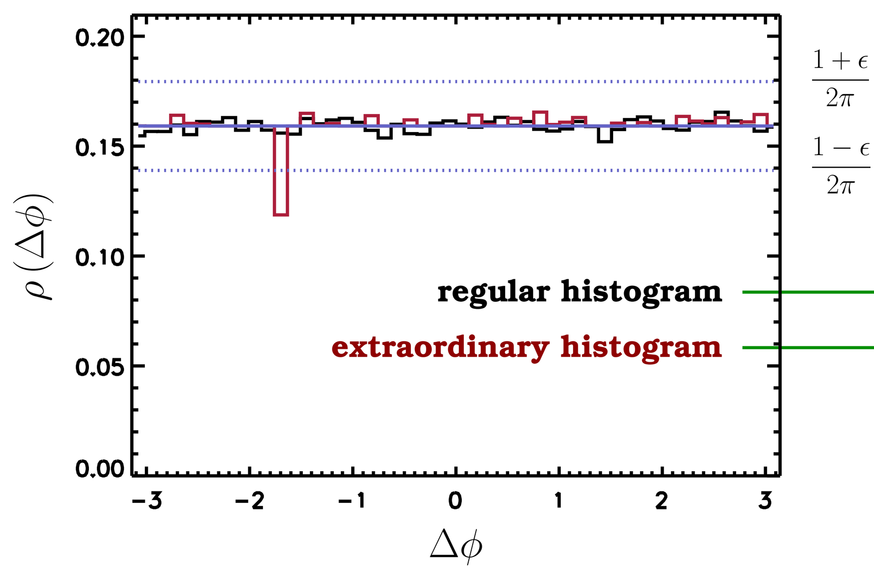
Finite number of measured phases → Finite number of phase increments n → Statistical noise on p-binned histograms $\tilde{\rho}(\Delta\phi)$

Mathematical formulation

Estimate an upper limit to the probability $\mathsf{P}\left(\tilde{\mathcal{Q}} > x\right)$ for $x > 0$, assuming a uniform underlying PDF of phase increments

Estimate the minimum value of $\tilde{\mathcal{Q}}$ assuring that $\mathcal{Q} \neq 0$

Regular and extraordinary histograms



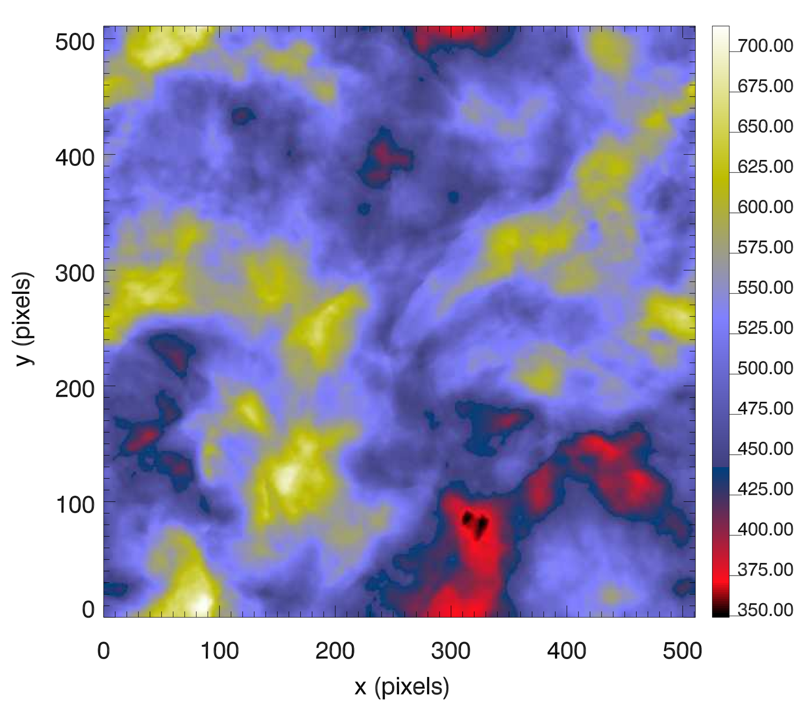
$$\mathsf{P}\left(\tilde{\mathcal{Q}} > x\right) \leqslant \underbrace{n\left[1 - \mathrm{Erf}\left(\epsilon\sqrt{\frac{p}{2(n-1)}}\right)\right]}_{\mathsf{P}_1} + \underbrace{\mathsf{P}\left(\chi^2 > \frac{2px(1-\epsilon)^2}{1+\epsilon}\right)}_{\mathsf{P}_2}$$

Adaptive upper limit estimation procedure

Choose values for P_1 and P_2 with $\mathsf{P}_1 \ll \mathsf{P}_2 \ll 1$ → Deduce ϵ from P_1, given n and p → Use quantiles of χ^2 to compute x

4 Phase structure quantity in simulated observations

Simulations of observations

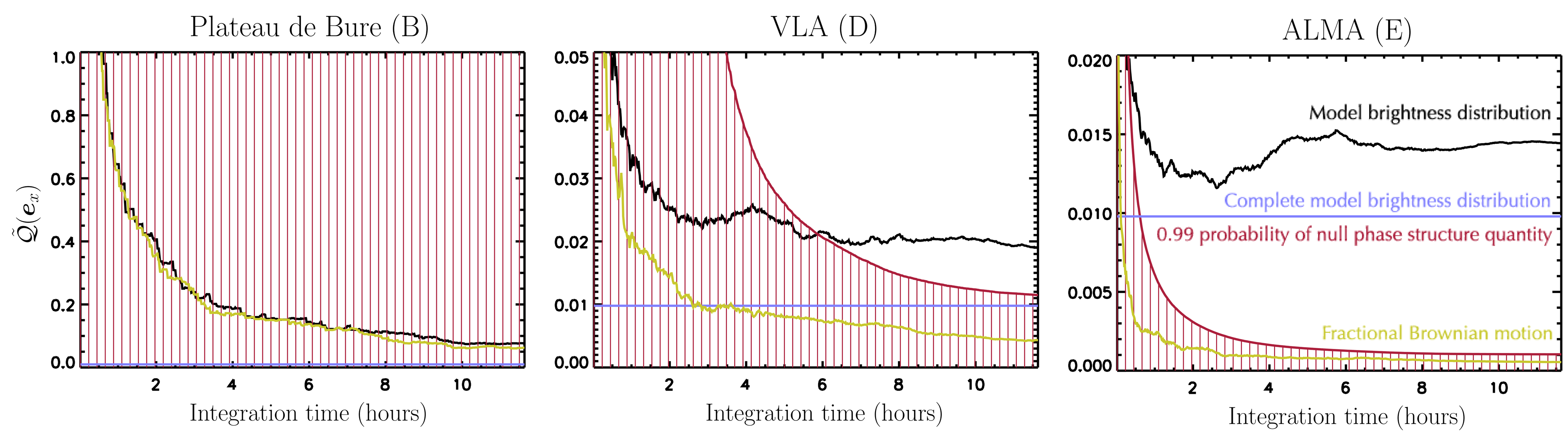


The column density of a compressible turbulence simulation (Porter *et al.*, 1994) is the model brightness distribution used as input to an interferometer simulator, the parameters of which are derived either from ALMA or from the existing arrays of Plateau de Bure and the VLA.

Instrument	ALMA	Bure	VLA
Antenna diameter (m)	12	15	25
Number of antennae	60	6	27
A configuration (m)	19 - 11527	32 - 400	807 - 37235
B configuration (m)	76 - 3005	71 - 331	247 - 11314
C configuration (m)	83 - 2303	48 - 229	79 - 3444
D configuration (m)	43 - 1618	24 - 113	41 - 1048
E configuration (m)	34 - 909	n.a.	n.a.
F configuration (m)	15 - 229	n.a.	n.a.

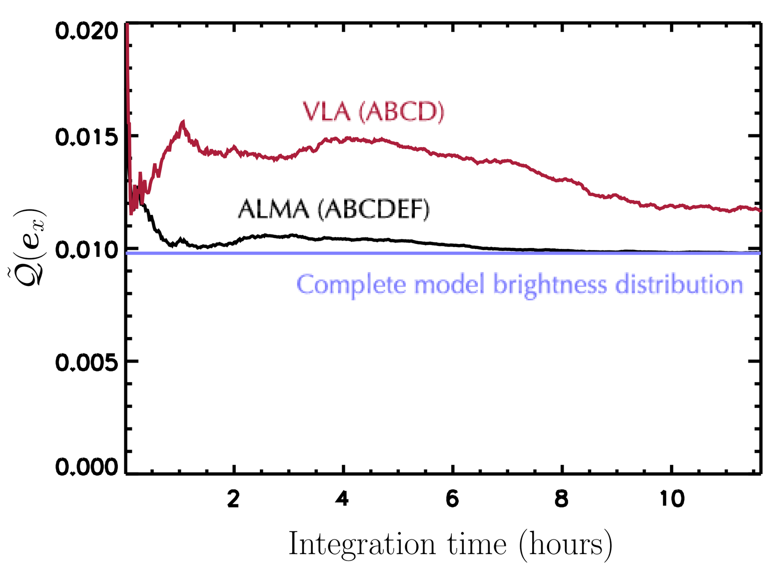
Wavelength	Longitude	Latitude	Dump time
1.3 mm	-67.75 °	-23.02 °	10 s

Single configuration and noise-free observations



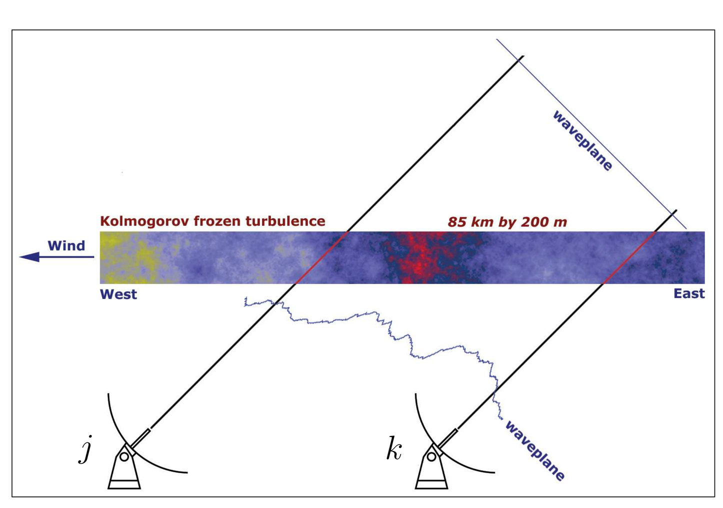
Evolution of the measured phase structure quantity $\tilde{\mathcal{Q}}$, for $\boldsymbol{\delta} = \boldsymbol{e}_x$ (unit vector along the k_x direction in Fourier space), with integration time. As the Earth rotates, more and more Fourier phases are measured, making the estimator $\tilde{\mathcal{Q}}$ of $\mathcal{Q}$ more reliable and the upper limit of $\tilde{\mathcal{Q}}$, given a uniform PDF of phase increments, drop. Significant detection of phase structure is possible with the VLA and ALMA, but not with Plateau de Bure.

Multiple configurations and noise-free observations



Evolution of the measured phase structure quantity $\tilde{\mathcal{Q}}(\boldsymbol{e}_x)$ with integration time, when multiple configurations are combined. Using the 6 configurations of ALMA, all Fourier components of the discretized model brightness distribution are measured after 7 hours. The phase structure quantity is then equal to that of the model field. No such thing is achieved using the VLA.

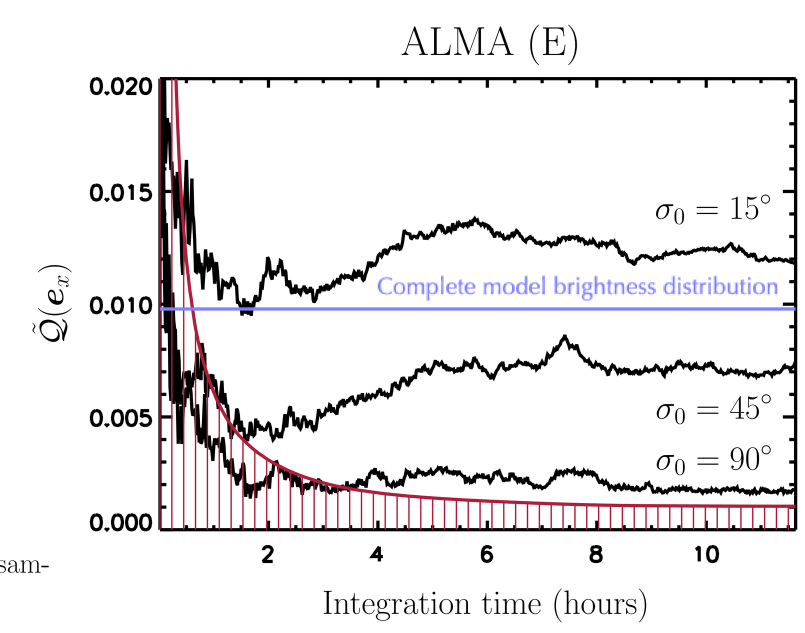
Single configuration observations in the presence of atmospheric phase noise



Atmospheric turbulence distorts incoming waveplanes by introducing phase noise $\phi_a(j,k,t)$, so that the measured visibilities (Fourier components) $V(u,v,t)$ are not the true visibilities $V_0(u,v)$.
The amount of atmospheric turbulence is given in terms of σ_0, which is the rms phase noise for a pair of antennae observing the zenith at $\lambda = 1.3$ mm, and separated by a baseline $d = 100$ m. At the Chajnantor site, σ_0 varies between $\sim 15°$ and $\sim 60°$ (Butler *et al.*, 2001).

$$\boxed{V(u,v,t) = \frac{1}{N(u,v,t)} \sum_{(j,k,t) \in (u,v)} V_0(u,v) \exp\left[i\phi_a(j,k,t)\right]}$$

$N(u,v,t)$: Number of cumulated samples at time t within cell (u,v)



5 Perspectives

- Keeping track of the phase measured by each baseline as a function of time, and computing phase increments along the baseline's track should markedly reduce contamination by atmospheric phase noise. In this approach, the lag vector $\boldsymbol{\delta}$ is no longer a control parameter, but a function of time and of the baseline.
- With high spectral resolution receivers, Fourier phase analysis may be applied to individual channel maps to study the structure of velocity fields.

REFERENCES

Butler, B.J., Radford, S.J.E., Sakamoto, S., & Kohno, K., ALMA Memo 365, 2001
Castellan, G., PhD Thesis, Université Paris-Sud, 2001 (`http://www.math.u-psud.fr/theses-orsay/2000/6039.html`)
Chiang, L.Y., & Coles, P., MNRAS, 311, 809, 2000
Porter, D.H., Pouquet, A., & Woodward, P.R., Physics of fluids, 6, 2133, 1994