SELECTED HYDRODYNAMICS PROBLEMS

Final Exam Année 2008-09

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Exercise 1.

1.) Show that the Euler equations of motion in spherical coordinates for the r and θ velocities are

$$\frac{\partial v_r}{\partial t} + (\boldsymbol{v\cdot\nabla})v_r - \frac{(v_\phi^2 + v_\theta^2)}{r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}$$

$$\frac{\partial v_\theta}{\partial t} + (\boldsymbol{v\cdot\nabla})v_\theta + \frac{(v_r v_\theta - v_\phi^2 \cot\theta)}{r} = -\frac{1}{\rho r}\frac{\partial P}{\partial \theta}$$

Recall

$$\boldsymbol{\nabla} = \boldsymbol{e_r}\frac{\partial}{\partial r} + \boldsymbol{e_\theta}\frac{\partial}{r\partial\theta} + \boldsymbol{e_\phi}\frac{\partial}{r\sin\theta\partial\phi}$$

Exercise 2.

2.) "Flying fish" can attain a velocity of about 15 m per sec as they leave the water, and can move through the air over distances of about 10 m. Adult flying fish have a mean size (characteristic dimension) L of 30 cm. The object of this execise is to see how much we may deduce from simple scaling law arguments, assuming that all fish lengths scale with L.

2a.) Show that in both water and air, the Reynolds number Re associated with these dimensions is large.

2b.) For large Re flow, motion at a velocity U creates an adverse pressure proportional to ρU^2. Under the assumption that the *power* required to overcome this pressure resistance (and dissipation) is proportional to the mass of the fish, show that $U \propto L^{1/3}$, that the velocity required to move the fish is proportional to the one-third power of its characteristic length.

2c.) In the air, are the fish "flying" (meaning that only a small fraction of the maximum possible lift force will support the fish), moving ballistically (maximum lift force always negligible), or gliding (maximum lift force could be significant)? Assume that a fish has a density equal to that of water, 1 g cm^{-3}.

Exercise 3.

3.) In a glass of water, the liquid surface can actually be a little higher than the highest part of the glass. A glass is usually circular, but to make the mathematics simple, our glass is shaped like a "U", as shown on page 6. The edges are at $x = \pm L$, the glass in infinite in the y direction, and we wish to find the height $\eta(x)$ above the surface $z = 0$, in a static equilibrium.

3a.) Show that η satisfies an equation of the form

$$A\eta - B\frac{d^2\eta}{dx^2} = C \tag{1}$$

where A, B, and C are constants. Evaluate A and B in terms of the surface tension γ, density ρ, and gravitational acceleration g. Show that the pressure P does not depend upon x and prove that $C = P$ at $\eta = 0$. Hence C is not known in advance.

3b.) Show that the solution to the differential equation, subject to the boundary condition $\eta(\pm L) = 0$ is

$$\eta(x) = h\left[\frac{\cosh(\alpha L) - \cosh(\alpha x)}{\cosh(\alpha L) - 1}\right] \tag{2}$$

where

$$\alpha^2 = \rho g/\gamma \tag{3}$$

and h is the maximum height of the water. Evaluate the "capillary length" $1/\alpha$. Plot $\eta(x)$ for the case $\alpha L >> 1$. (Your plot need not be exact, but it should show important qualitative features.)

Exercise 4.

4.) A viscous flow is present in a region $R > a$, where R and ϕ are cylindrical coordinates. There is a cylinder at $R = a$ rotating at angular velocity Ω, which induces a rotation velocity $v_\phi(R)$ in the fluid. It also provides a uniform suction ("aspiration," en français) that induces an inward radial velocity $v_R(R)$. $v_R = -U$ at $R = a$. Therefore, the flow passes directly through the cylinder in the radial direction, but satisfies the no-slip boundary condition $v_\phi(a) = a\Omega$.

4a.) Prove that $v_R(R) = -Ua/R$.

4b.) Prove that (be careful with ∇^2!):

$$R^2 \frac{d^2 v_\phi}{dR^2} + (\mathcal{Q}+1) R \frac{d v_\phi}{dR} + (\mathcal{Q}-1) v_\phi = 0 \tag{4}$$

where $\mathcal{Q} = Ua/\nu$ is the Reynolds number at the cylinder.

4c.) Solve this equation exactly together with the boundary condition at $R = a$. Show that if $\mathcal{Q} < 2$, there is a unique solution with finite circulation $2\pi R v_\phi$ as $R \to \infty$, but that if $\mathcal{Q} > 2$ the solution is not unique! The Navier-Stokes solution does *not* necessarily have a unique solution for a given set of boundary conditions.

Exercise 5.

5. Describe the phenomenon of "boundary layer separation." What causes it? Why is it dangerous for aircraft?

USEFUL RESULTS.

Spherical unit vectors:

$$\boldsymbol{e_r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\boldsymbol{e_\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$\boldsymbol{e_\phi} = (-\sin\phi, \cos\phi, 0)$$

Nonvanishing derivatives of spherical unit vectors:

$$\frac{\partial \boldsymbol{e_r}}{\partial\theta} = \boldsymbol{e_\theta}$$
$$\frac{\partial \boldsymbol{e_r}}{\partial\phi} = \sin\theta\,\boldsymbol{e_\phi}$$
$$\frac{\partial \boldsymbol{e_\theta}}{\partial\theta} = -\boldsymbol{e_r}$$
$$\frac{\partial \boldsymbol{e_\theta}}{\partial\phi} = \cos\theta\,\boldsymbol{e_\phi}$$
$$\frac{\partial \boldsymbol{e_\phi}}{\partial\phi} = -(\sin\theta\,\boldsymbol{e_r} + \cos\theta\,\boldsymbol{e_\theta}) = -\boldsymbol{e_R}$$

Surface tension of water $\simeq 0.07$ J m^{-2}. $\eta(\text{gas}) \simeq 2\times10^{-5}$ kg m^{-1} s^{-1}.
$\rho(\text{air}) = 1.2$ kg m^{-3}. $\nu(\text{water}) = 10^{-6}$ m^2 s^{-1}. $\eta(\text{water}) = 10^{-3}$kg m^{-1} s^{-1}.

$$\omega = -\frac{2\Omega\sin\theta}{r}\frac{k_y}{k^2},\ dx = r d\theta,\ dy = r\sin\theta\, d\phi$$

$$\rho\left(\frac{\partial v_R}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R - \frac{v_\phi^2}{R}\right) = -\boldsymbol{\nabla}P + \eta\left(\nabla^2 v_R - \frac{v_R}{R^2} - \frac{2}{R^2}\frac{\partial v_\phi}{\partial\phi}\right)$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\phi + \frac{v_\phi v_R}{R}\right) = -\boldsymbol{\nabla}P + \eta\left(\nabla^2 v_\phi + \frac{2}{R^2}\frac{\partial v_R}{\partial\phi} - \frac{v_\phi}{R^2}\right)$$

$$\frac{1}{R}\frac{\partial(Rv_R)}{\partial R} + \frac{1}{R}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} = 0$$