Exercise 1.

1. Magnetic fields align small dust grain along their lines of force, causing starlight to be polarized perpendicular to the field line—the other component is absorbed. In this way, patterns of magnetic field lines can be "seen" in the sky. Consider an Alfvén wave which causes a field line displacement of the form

$$y = a\cos(kx - \omega t)$$

1a.) y cannot be directly measured. But $y' = \partial y/\partial x$ can be measured by looking at the polarization patterns of starlight. Show that this quantity is independent of the distance to the source.

The quantity $\partial y/\partial x$ is the ratio of two distances, and therefore is itself independent of distance: the "scale factor" divides out of the ratio. Explicity, y' is ka times a sine function, and k is proportional to one over the distance, while a is proportional to the distance.

1b.) $\dot{y} = \partial y/\partial t$ can also be measured, since the gas motion follows the magnetic field lines, and the gas velocity can be directly measured. Knowing only y' and $\dot{y}$ and the density of the gas, explain in detail how the average magnetic field strength can be determined. (This idea is due to Chandrasekhar and Fermi.)

The ratio of $(y'/\dot{y})^2$ is k^2/ω^2, which for this class of wave is $1/v_A^2$, where v_A is the Alfvén velocity. This is proportional to ρ/B^2, where ρ is the density and B is the magnetic field strength. Knowing ρ then allows B to be determined.

1c.) Does this method work even if y is not a single cosine, but a sum of many cosines of different wavenumbers?

Yes! If you average both $(y')^2$ and $(\dot{y})^2$ over distance, then form their ratio as above, this ratio will be the sum over $(a_i k_i)^2$ in the numerator and $(a_i\omega_i)^2$ in the denominator. Here, the index i labels each wavenumber, frequency, and amplitude that is present. Since $\omega_i = k_i v_A$, the ratio reduces to $1/v_A^2$, as before.