Formation Interuniversitaire de Physique

M2
Magnetohydrodynamics

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Contents

1 Fundamentals

1.1 Opening Comment

The behavior of a gas subject to large-scale gravitational and magnetic forces is enormously rich and full of surprises. One of my goals in giving this course is to try to give you, the student encountering the topic for the first time, a sense of both the generality and the depth of the problems we are struggling with. Truly, there is not an area of modern astrophysics that is not touched in some way by the dynamical behavior of gases. Astrophysical gas dynamics is, in the view of your author, the most fundamental component of astrophysics. It is impossible to understand star formation, stellar structure, planet formation, accretion disks, or anything in the early universe without a detailed knowledge of the dynamics of magnetizedgases. So why don't we start?

1.2 Governing Equations

Although the fundamental objects are the atomic particles that comprise our gas, we shall work in the limit in which the matter is regarded as a *nearly* continuous fluid. The fact that this is not *exactly* a continuous fluid manifests itself in many ways, the most important of which is the equation of state of an ideal gas, which depends upon the notion of rapid atomic collisions separated by "long" intervals of time when the atoms are, in essence, free. But more subtle transport effects are also present, like viscosity and thermal conduction, both of which are a consequence of atomic collisions.

One of the most interesting and salient features of astrophysical gases is that they are almost always magnetized. This allows modes of behavior that are absent in an ordinary nonmagnetized gas (e.g. shear waves). This sometimes has profound consequences, especially in rotating systems. The dynamics of magnetized gases is known as magnetohydrodynamics, or MHD for short. The ohmic resistivity of a magnetized gas is another example of a collisional process involving individual particles; in this case one of the particles must be the current carrying electrons.

I shall assume that the reader is familiar with the basic equations of standard hydrodynamics. If not, they[1] may review a standard textbook (my favorite is *Elementary Fluid Dynamics* by D.J. Acheson), or the set of extensive notes I have prepared for my course Hydrodynamics, Instability and Turbulence. We begin with a very brief review.

[1] In these notes, I will use "they" to mean generically "he or she".

1.2.1 Mass Conservation

The statement of mass conservation is expressed by the equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{1}$$

Here ρ is the mass density and $\boldsymbol{v}$ is the velocity field. The content of this equation is simply that if there is net a mass flux into or out of a fixed volume, the mass within that volume must change accordingly. If the flow happens to be divergence free, the density of an individual fluid element remains constant, and if all fluid elements start with the same density, the density remains everywhere constant.

1.2.2 Newtonian Dynamics

Our second fundamental equation is a statement of Newton's second law of motion, that applied forces cause acceleration in a fluid. The acceleration refers to an individual element of fluid, hence the time derivative is expressed as a *total* derivative, following the path of the element:

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \right] = \boldsymbol{F} \tag{2}$$

where the right side is the sum of the forces on the fluid element. The combination D/Dt arises often, and it is called the *Lagrangian derivative.*

A fundamental force that is always present acting on a fluid is the pressure. We shall be working with gases obeying the ideal gas equation of state, and the pressure is then given by

$$P = \frac{\rho k T}{m} \tag{3}$$

where T is the temperature in Kelvins, k is the Boltzmann constant 1.38×10^{-23} J K^{-1}, and m is the mass per particle. For a fully ionized gas consisting of protons and electons, m is $0.5 m_p$, one half of the proton mass (the electron mass being negligible in comparison). The quantity kT/m arises often enough that it will be given its own name:

$$c_S^2 \equiv \frac{kT}{m} \tag{4}$$

where the subscript S refers to "sound" for reasons that will become clear later.

The pressure arises from the kinetic energy of the gas particles themselves, and these particles must never be confused with fluid elements. A fluid element is small enough that it has uniquely defined dynamic and thermodynamic attributes (e.g. velocity and pressure), but large enough to contain a vast number of particles. A fluid element has a well-defined entropy for example, an atom does not.

There is a very simple relationship between the pressure P and internal energy density $\mathcal{E}$ of an ideal gas:

$$\mathcal{E} = \frac{P}{\gamma - 1}. \tag{5}$$

Here γ is the adiabatic index of the gas. It is equal to 5/3 for single particles, and 7/5 for diatomic molecules.

A pressure exerts a force only if it is not spatially uniform. For example, the pressure force in the x direction on a slab of thickness dx and area $dy\,dz$ is

$$[P(x - dx/2, y, z, t) - P(x + dx/2, y, z, t)]dy\,dz = -\frac{\partial P}{\partial x} dV \tag{6}$$

There is nothing special about the x direction, so the force per unit volume from a pressure is more generally $-\boldsymbol{\nabla} P\,dV$.

Other forces can be added on as needed. One force of obvious importance in astrophysics is gravity. The Newtonian gravitational acceleration $\boldsymbol{g}$ can always be derived from a potential function

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi \tag{7}$$

If the field is from an external source, then Φ is a given function of $\boldsymbol{r}$ and t, otherwise it must be computed along with the evolution of the fluid itself. We shall discuss the problems of self-gravity later in the course.

Another force that we must consider, which will be front and center in this course, arises from the presence of a magnetic field. As we have already noted, magnetic fields allow a gas to behave in ways not allowed when the field vanishes, and the additional degrees of freedom imparted to a gas mean that magnetic forces can be very important even when the field *appears* to be weak! To calculate the magnetic force per unit volume exerted by a magnetic field, start with the Maxwell equation

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0 \boldsymbol{J} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{8}$$

The effects of the displacement current are negligible for nonrelativisitic fluids, since they involve time delays associated with light propagation. Hence, the current density is determined by the magnetic field geometry:

$$\boldsymbol{J} = (1/\mu_0)\boldsymbol{\nabla}\times\boldsymbol{B} \tag{9}$$

The Lorentz force per unit volume is $\boldsymbol{J}\times\boldsymbol{B}$, assuming that the gas is everywhere locally neutral.

In the absence of dissipational processes, the equation of motion for a magnetized gas is therefore

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\rho\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} \tag{10}$$

1.2.3 Energetics

The thermal energy behavior of the gas is described by the internal energy loss equation, which is most conveniently expressed in terms of the entropy per particle. The entropy is defined up to an (unimportant) additive constant, and is given by

$$s = \frac{S}{N} = \frac{k}{\gamma - 1}\ln P\rho^{-\gamma} \tag{11}$$

where N is the number of particles, γ is the adiabatic index (equal to $1+2/f$ where f is the number of degrees of freedom of a particle).

Exercise. Derive the above expression from $dE = -PdV + TdS$, $P = \rho kT/m = (\gamma - 1)\mathcal{E}$, $E = \mathcal{E}V$.

The entropy of a fluid element is conserved unless there is a loss or gain of heat from radiative processes or from dissipation. If n is the number of particles per unit volume, then

$$nT\frac{Ds}{Dt} = \frac{P}{\gamma - 1}\frac{D\ln P\rho^{-\gamma}}{Dt} = \text{volume heating rate} \equiv \dot{Q} \tag{12}$$

If there are no radiative losses or gains and no dissipation, as is often the case when the fluid motions are too rapid for heat to escape, the fluid is said to be *adiabatic* and the right side of the above is zero. Note that the internal thermal energy is *not* conserved in an adiabatic fluid because of compression or expansion. As an exercise, the reader should show that c_S^2 satisfies the equation

$$\rho\frac{D}{Dt}\frac{c_S^2}{\gamma - 1} = -P\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{13}$$

for an adiabatic gas. (Use the entropy and mass conservation equations.) The temperature of a fluid element, like the density, remains fixed only if the motions are incompressible.

1.3 The vector "v dot grad v"

The vector $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is more complicated than it appears. In Cartesian coordinates, matters are simple: the x component is just $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_x$, and similar for y, z. But in cylindrical coordinates, say, the radial component of this vector is NOT $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R$, where v_R is the radial velocity component. Rather, we must take care to write

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\nabla}(v_R\boldsymbol{e_R} + v_\phi\boldsymbol{e_\phi} + v_z\boldsymbol{e_z}) \tag{14}$$

where the $\boldsymbol{e_i}$ are unit vectors in their respective directions. In Cartesian coordinates, these unit vectors would be constant, but in any other coordinate system they change with position. You should be able to show that

$$\frac{\partial \boldsymbol{e_R}}{\partial\phi} = \boldsymbol{e_\phi}, \quad \frac{\partial \boldsymbol{e_\phi}}{\partial\phi} = -\boldsymbol{e_R}, \tag{15}$$

and that there are no other unit vector derivatives in cylindrical coordinates. (Do it now. Hint: $\boldsymbol{e_R} = \cos\phi\boldsymbol{e_x} + \sin\phi\boldsymbol{e_y}$, and $\boldsymbol{e_\phi} = -\sin\phi\boldsymbol{e_x} + \cos\phi\boldsymbol{e_y}$.) Thus, the radial component of $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_R - \frac{v_\phi^2}{R}, \tag{16}$$

and the azimuthal component is

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_\phi + \frac{v_R v_\phi}{R} \tag{17}$$

The extra terms are related to centripetal and Coriolis forces, though more work is needed to extract the latter...a piece of it still remains in the gradient term!

1.4 Rotating Frames

It is often useful to work in a frame rotating at a constant angular velocity Ω, perhaps the frame in which an orbiting planet appears at rest around its star. The same rule that applies to ordinary point mechanics applies here as well: add

$$-2\boldsymbol{\Omega}\times\boldsymbol{v} + R\Omega^2\boldsymbol{e_R} \tag{18}$$

to the applied forces operating on a fluid element (the right side of the MHD equation of motion). The first term is the Coriolis force, the second is the centrifugal force, $\boldsymbol{\Omega}$ is in the vertical direction, and all velocities are measured relative to the rotating frame of reference.

1.5 Manipulating the Fluid Equations

For a particular astrophysical problem, working in cylindrical or spherical cooridinates is often the most convenient, but for proving general theorems or identities, Cartesian coordinates are usually the simplest to use. In this case, there is a formalism that makes working with the MHD fluid equations much easier.

The index i, j, or k will represent Cartesian component x, y, or z. Hence v_i means the ith component of v, which may any of the three depending upon what value i is chosen. So v_i is a way to write $\boldsymbol{v}$. The gradient operator $\boldsymbol{\nabla}$ is written ∂_i, in a way that should be self-explanatory.

Next, if an index appears twice, it is understood that it is to be summed over all the values x, y, and z. Hence

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i = A_x B_x + A_y B_y + A_z B_z, \tag{19}$$

and

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = (v_i \partial_i) v_j \tag{20}$$

In this last example i is a "dummy index": the actual vector component is represented by j. The dynamical equation of motion in this notation is

$$\rho[\partial_t + (v_i \partial_i)] v_j = -\partial_j P - \rho \partial_j \Phi \tag{21}$$

Sometimes the "rot" (or "curl") operator is needed. For this, we intoduce the Levi-Civita symbol ϵ^{ijk}. It is defined as follows:

- If any of the i, j, or k are equal to one another, then $\epsilon^{ijk} = 0$.
- If $ijk = 123$, 231, or 312, the so-called even permutations of 123, then $\epsilon^{ijk} = +1$.
- If $ijk = 132$, 213, or 321, the so-called odd permutations of 123, then $\epsilon^{ijk} = -1$.

By explicitly writing out each side of the equation, it is straightforward to show that

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \epsilon^{ijk} \partial_i A_j. \tag{22}$$

Here, the vector component is represented by the index k. Don't forget to sum over i and j! ϵ^{ijk} is of course used in the ordinary cross product as well:

$$\boldsymbol{A} \times \boldsymbol{B} = \epsilon^{ijk} A_i B_j. \tag{23}$$

Notice that

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \epsilon^{ijk}A_kB_iC_j \tag{24}$$

which proves that any even permutation of the vectors on the left side of the equation must give the same value, and an odd rearrangement gives the same value with the opposite sign.

A double cross product looks complicated:

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \epsilon^{lkm}A_l(\epsilon^{ijk}B_iC_j) = \epsilon^{mlk}\epsilon^{ijk}A_lB_iC_j. \tag{25}$$

The last equality follows because mlk is an even permutation of lkm. This looks unpleasant, but fortunately there is an identity that saves the day:

$$\epsilon^{mlk}\epsilon^{ijk} = \delta_{mi}\delta_{lj} - \delta_{mj}\delta_{li} \tag{26}$$

where δ_{ij} is the Kronecker delta function (equal to zero if i and j are different, unity if they are the same). One may always prove this by brute force, but an outline of a shorter proof would be to note that the left side has at most one nonvanishing term in its sum, under all circumstances. Moreover, for this one term not to vanish, the index pair (i,j) must be the same *distinct* pair of numbers as (m,l) or (l,m). You can now check that in all cases, the sign $+1$ or -1 always comes out correctly on both sides of the equation. With this identity, our double cross product becomes

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = B_mA_jC_j - C_mA_iB_i = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}). \tag{27}$$

Our final example is to derive an expression for

$$\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \epsilon^{ijk}A_i(\epsilon^{lmj}\partial_lB_m) = \epsilon^{kij}\epsilon^{lmj}(A_i\partial_lB_m) \tag{28}$$

Using our identity (26), this becomes

$$(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(A_i\partial_lB_m) = A_i\partial_kB_i - A_i\partial_iB_k = A_i\partial_kB_i - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{29}$$

One consequence of this is a representation of $A_i\partial_kB_i$ in any coordinate system:

$$A_i\partial_kB_i = \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{30}$$

Another *particularly* important application of (30) is to the Lorentz force expression, something very important for this course. Substituing $\boldsymbol{B}$ for $\boldsymbol{A}$ in the above gives us:

$$(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{1}{2}\boldsymbol{\nabla}B^2 + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{31}$$

The first term on the right side has the form of a magnetic pressure gradient; the second behaves like a tension force. It depends on the derivative of $\boldsymbol{B}$ along its length, and if the magnitude of $\boldsymbol{B}$ remains fixed, the force must be perpendicular to $\boldsymbol{B}$ itself. The effect of this tension force is profound, allowing a magnetized gas to support shear waves (known as Alfvén waves) that do not exist in a standard, nonmagnetized fluid. In this sense, a magnetized gas behaves more like a solid!

1.6 The Conservation of Vorticity

Let us return, just for the moment, to an unmagnetized fluid. We start with the following identity, which follows immediately from the results of the previous section:

$$\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = \frac{1}{2}\boldsymbol{\nabla} v^2 - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \tag{32}$$

Using this result to replace $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ in the dynamical equation of motion results in

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla} v^2 - \boldsymbol{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi \tag{33}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ is known as the vorticity. If we take the curl of this equation and remember that the curl of the gradient vanishes, we find

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}P) \tag{34}$$

Let us once again consider the case where either ρ is constant, or when P is a function only of ρ. In that case, the right hand side vanishes and:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0. \tag{35}$$

To understand what this means, consider a closed linear curve, like a ring, moving with the fluid. The integral $\int \boldsymbol{v} \cdot \boldsymbol{dl}$ around the ring is also $\int \boldsymbol{\omega} \cdot \boldsymbol{dA}$, taken over an area that is bounded by the ring. This is the vorticity flux. How does the vorticity flux change with time as the fluid evolves?

Let us consider more generally a generic equation of the form

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) + \boldsymbol{\nabla}\Phi \tag{36}$$

where Φ is a potential function. The curl of this equation leads directly to equation (35) for the special case $\boldsymbol{A} = \boldsymbol{v}$, but it is better to retain generality here, because this same equation will be useful when we investigate the behavior of magnetic fields. Expanding the double cross product on the right (which you should be able to do by now!) and regrouping leads to

$$\frac{DA_i}{Dt} = v_j \partial_i A_j + \partial_i \Phi \tag{37}$$

where D/Dt is the standard Lagrangian derivative, and we have, of course, switched over to index notation. We now consider the change in the line integral of the vector field $\boldsymbol{A}$ over a closed curve moving with the fluid itself:

$$\frac{D}{Dt}\oint \boldsymbol{A}\cdot\boldsymbol{dl} = \oint\left[\frac{D\boldsymbol{A}}{Dt}\cdot\boldsymbol{dl} + \boldsymbol{A}\cdot\frac{D\boldsymbol{dl}}{Dt}\right] \tag{38}$$

Hmmm. How interesting. We are taking the derivative of a differential $\boldsymbol{dl}$. Have you ever done that before? Don't panic. The Lagrangian change of an embedded line element $\boldsymbol{dl}$ moving with the fluid is just the difference between the velocities at each of the two endpoints of the segment $\boldsymbol{dl}$, multiplied by a time interval. To be precise, if $\boldsymbol{dl}$ is the line element at time $t=0$, and $\boldsymbol{dl'}$ is the same line element an instant later at time $t=\Delta t$, then

$$\boldsymbol{dl'} = \boldsymbol{dl} + [\boldsymbol{v}(\boldsymbol{r}+\boldsymbol{dl},t) - \boldsymbol{v}(\boldsymbol{r},t)]\,\Delta t = \boldsymbol{dl} + \Delta t\ (\boldsymbol{dl}\cdot\boldsymbol{\nabla})\boldsymbol{v}, \tag{39}$$

or

$$\frac{D\boldsymbol{dl}}{Dt} = \frac{\boldsymbol{dl'}-\boldsymbol{dl}}{\Delta t} = (\boldsymbol{dl}\cdot\boldsymbol{\nabla})\boldsymbol{v} \tag{40}$$

as $\Delta t \to 0$. This may also be written

$$\frac{Ddl_j}{Dt} = dl_i\partial_i v_j \tag{41}$$

We then have from equation (37)

$$\frac{D\boldsymbol{A}}{Dt}\cdot\boldsymbol{dl} = dl_i\ v_j\partial_i A_j + dl_i\partial_i\Phi \tag{42}$$

and

$$\boldsymbol{A}\cdot\frac{D\boldsymbol{dl}}{Dt} = A_j dl_i\partial_i v_j \tag{43}$$

Adding these last two equations gives

$$\oint\frac{D}{Dt}(\boldsymbol{A}\cdot\boldsymbol{dl}) = \oint dl_i\partial_i(\Phi + v_jA_j) = \oint \boldsymbol{dl}\cdot\boldsymbol{\nabla}(\Phi+v_jA_j) \tag{44}$$

This is a perfect gradient function integrated around a closed curve. Since the beginning and end points are the same, it must vanish. The line integral $\oint \boldsymbol{A}\cdot\boldsymbol{dl}$ is conserved with the fluid. In particular, when $\boldsymbol{A}=\boldsymbol{v}$, the velocity circulation integral along with the vorticity flux surface integral are conserved in the Lagrangian sense, moving with the fluid. We shall see very soon that the same is true for the magnetic field and magnetic flux.

The fact that the integral $\oint \boldsymbol{v}\cdot\boldsymbol{dl}$ around any closed curve in the fluid remains constant as the it flows with the fluid is known as vorticity conservation. Another way to say the same thing is that the field lines of vorticity

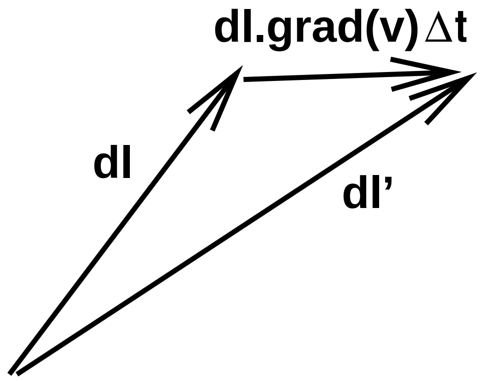

Figure 1: The change of a line element $\boldsymbol{dl}$ in time Δt.

$\boldsymbol{\omega}$ are "frozen" into the fluid. Once again, this is not a completely general fluid result, even if there is no magnetic field. We had to assume either that ρ is constant, or that P and ρ are functionally related, $P = P(\rho)$. (This is called a *barotropic* fluid.)

With the help of our $\epsilon^{ijk}\epsilon^{lmk}$ identity and just a little work, it is quite straightforward to show that the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0 \tag{45}$$

is the same equation as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = +(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\omega}\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{46}$$

Now, mass conservation implies

$$\frac{D\ln\rho}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{47}$$

so that our equation becomes

$$\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\frac{D\ln\rho}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v}, \tag{48}$$

or

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{1}{\rho}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} \tag{49}$$

This is a very interesting result! Notice that $\boldsymbol{\omega}/\rho$ satisfies exactly the same equation (41) as the line element $\boldsymbol{dl}$. But, by definition, $\boldsymbol{dl}$ was a small line segment moving with the fluid. So this is a direct way of showing that $\boldsymbol{\omega}/\rho$ moves with the fluid.

Next, consider a flow that is strictly two-dimensional, with nothing depending on the vertical coordinate z, and no z component of the velocity. Then, in the mass conservation equation (47), it makes sense to multiply by ρ and integrate over z. This has the very simple effect of replacing the density ρ with the integrated "column density" Σ in the analysis that follows. Also, since ω now has *only* a z component, the right side of equation (49) must vanish! We then find instead of (49),

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\Sigma}\right) = 0 \tag{50}$$

This is known as the conservation of potential vorticity. It is an extremely useful and powerful constraint in the study of two-dimensional turbulence, as well as in studying an important class of disturbances in planetary atmospheres known as Rossby waves.

Exercise. Consider purely rotational flow, with the velocity $\boldsymbol{v}$ having only a ϕ component v_ϕ. In general, v_ϕ could depend upon R and z, but show that if vorticity conservation holds, then under steady conditions v_ϕ cannot depend upon z. This is known as *von Zeippel's* theorem.

Exercise. Two-dimensional turbulence in a fluid is never spontaneous, it must always be driven externally. This is *not* true of three dimensional turbulence. Explain this far reaching result in terms of potential vorticity conservation. (Hint: What would happen if we had even a tiny amount of dissipation in a two-dimensional fluid?)

Exerccise. In planetary atmospheres, local disturbances that lose vorticity find their way up to the north pole (or down to the south pole), and settle down as "polar vortex rings." Explain. (Hint: The total vorticity of a disturbance, including the contribution from the planet's rotation plus the intrinsic vorticity within the gas, must be conserved.)

2 Magnetohydrodynamics (MHD)

2.1 Magnetic Forces

We return to magnetic fields. Astrophysical gases are almost always at least partially ionized. This is not too surprising: a glass of distilled water is ionized at the level of one part in 10^7, and salty sea water is much more ionized: it is a very good conductor. A medium can be almost entirely neutral and still behave like a good conductor. All but the coolest and densest astrophysical gases (e.g., protostellar disks) are electrodynamically active.

The Lorentz force per unit volume in the gas is

$$\boldsymbol{F} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \tag{51}$$

where ρ_e is the charge density, $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the current density, and $\boldsymbol{B}$ is the magnetic field. The gases of interest here are all electrically neutral, so that $\rho_e = 0$. This means that the only part of the Lorentz force that affects the gas is the magnetic part.

We have already encountered the Lorentz force in our discussion of the equation of motion for a magnetized gas:

$$\boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \tag{52}$$

In the last section, we showed that

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = -\frac{1}{2} \boldsymbol{\nabla} B^2 + (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} \tag{53}$$

Thus, the dynamical equation of motion for a magnetized gas is

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla} \left(P + \frac{B^2}{2\mu_0} \right) - \rho \boldsymbol{\nabla} \Phi + \left(\frac{\boldsymbol{B}}{\mu_0} \cdot \boldsymbol{\nabla} \right) \boldsymbol{B} \tag{54}$$

The first magnetic term on the right clearly behaves like a sort of pressure. Magnetic fields lines of force do not like to be squeezed any more than gas molecules do.

The $(\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B}$ term is less obvious. We have noted that it corresponds to a sort of magnetic tension. Notice that it vanishes when the magnetic field does not change along the direction in which the field line itself is oriented. On the other hand, when there are such changes, and the field line is bent, the resulting force acts in the direction of restoring the field line back to an unbent position. In fact, this can be made quantitative: there is a magnetic analogue to waves propagating along an ordinary string that is under tension. In the case of “magnetic strings,” these waves are called Alfvén waves.

2.2 Induction Equation

Having introduced the magnetic field, we need to know how it evolves when there are changes in the fluid. The magnetic field adds one more variable to our problem (well, three actually, since there are three components of $\boldsymbol{B}$), so that we need some more equations. The motion of the gas causes charged particles to move relative to one another, and the resulting electrical currents in turn generate new magnetic fields. These affect the currents, that change the fields again, that ... Help. It seems like a complicated mess!

Fortunately there is indeed help in the form of a great simplifying principle: in a pefect conductor, the electric field vanishes. Actually, what we need to say is that in the rest frame of the conductor, the electric field vanishes. In a frame in which the conductor (in our case the conducting gas fluid element) moves, the total Lorentz force, not the electric field, must vanish. In other words,

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0. \tag{55}$$

So even though we have assumed conditions for charge neutrality, there must be an electric field! Hmmm. Wait. If the divergence of this electric field does not vanish, then according to Maxwell (or Coulomb!) there must be a local charge density, and then charge neutrality cannot hold. This certainly looks like looks like a contradiction. Well, guess what? The divergence of the electric field does *not,* in general, vanish. In a moment, we'll come back and explain why this is not really a contradiction, but for the time being let us continue as though we have nothing to worry about.

Faraday's law of induction is

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} \tag{56}$$

and with $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$, this becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{57}$$

This is the equation we need to determine the magnetic field. By knowing how the spatial gradients of $\boldsymbol{B}$ are behaving, we may compute how the field evolves in time, thanks to the powerful constraint that the Lorentz force on the charge carriers must vanish.

Notice something quite remarkable: the magnetic field satisfies the same equation as the vorticity. In particular, equation (57) can be recast in the form of equation (36), by "uncurling" it! That means everything we learned about vorticity, in particular that it is frozen in to the fluid, also holds for the magnetic field. *Magnetic flux, $\int \boldsymbol{B} \cdot \boldsymbol{dA}$, is conserved as the area moves with the fluid.* But unlike the case of vorticity conservation, which depended upon a restrictive relationship between P and ρ, magnetic flux conservation depends only upon there being no dissipation (i.e., electrical resistance) in the gas. This is generally an excellent approximation.

2.3 Self-consistency

Why don't we have a contradiction with the fact that $\nabla \cdot \boldsymbol{E}$ is not zero? The answer is that while not zero, it is in fact, you know, small. Small?? Don't give me that. That answer is not good enough. How small? Very small indeed: of order v^2/c^2 (c is the speed of light). This, as we will see, is precisely of the same order as the neglected displacement current.

To estimate $\nabla \cdot (\boldsymbol{v} \times \boldsymbol{B})$, assume that any magnetic field gradients are as large as they can be (of order $\mu_0 J$), and that J is also as large as it can be, of order the ion charge density times v, $\rho_i v$ (the current density could be much smaller, since it is proportional to the *difference* between ion and electron velocities). Then

$$\nabla\cdot(\boldsymbol{v}\times\boldsymbol{B}) \sim v\mu_0 J \sim \frac{\rho_i}{\epsilon_0}\frac{v^2}{c^2}. \tag{58}$$

That answer, that the divergence of the electric field is of order v^2/c^2 times the ion charge density, really *is* good enough. Not only is it permitted to ignore the divergence of the electric field, it is required! We have already not included the displacement current, and this too is a correction of order v^2/c^2. In this case, if L is a characteristic length and $\partial/\partial t \sim v/L$, then

$$\epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} \sim \epsilon_0\mu_0\frac{vE}{L} \sim \epsilon_0\mu_0\frac{v^2 B}{L} \sim \epsilon_0\mu_0^2 v^2 J \tag{59}$$

which is indeed of order $(v^2/c^2)\mu_0 J$. Corrections of order v^2/c^2 are truly relativistic, and we must ignore them to be self-consistently *nonrelativistic.*

A Summary of the Dissipationless Equations of Motion

From now on, we shall drop the subscript "0" on μ_0, and write μ.

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0 \tag{60}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\boldsymbol{v} = -\nabla\left(P + \frac{B^2}{2\mu}\right) - \rho\nabla\Phi + \frac{1}{\mu}(\boldsymbol{B}\cdot\nabla)\boldsymbol{B} \tag{61}$$

$$\frac{P}{\gamma - 1}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\nabla\right)\ln P\rho^{-\gamma} = 0 \tag{62}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) \tag{63}$$

3 Fundamentals

In this section, a detailed derivation of the fundamental MHD equations is presented. The discussion will be more technical here than in most of the rest of the course, but it is very important to see how the basic governing equations of the subject arise, and much of this material is not so easy to find outside of specialized treatments. I hope the reader will have the patience to read carefully through this section.

In astrophysics, we are very often interested in the MHD behavior of a gas that is almost entirely neutral. This may seem like contradictory, since a neutral gas has no charge carriers, but the key word is "almost." Even a very small population of charge carriers will make the gas magnetized, as we will shortly see.

A typical environment is a gas cloud consisting of neutral particles (predominantly H_2 molecules), electrons, and ions. Each species (denoted by subscript s) is separately conserved, and obeys the mass conservation equation

$$\frac{\partial \rho_s}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho_s \boldsymbol{v_s}) = 0 \tag{64}$$

where ρ_s is the mass density for species s and $\boldsymbol{v_s}$ is the velocity. The symbols of the flow quantities (e.g. $\boldsymbol{v}$, ρ, etc.) for the dominant neutral species will henceforth be presented without subscripts.

So far, everything is simple. The dynamical equations become more coupled, however, since we need to include interactions between the different species. The dynamical equation for the neutral particles is

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi - \boldsymbol{p_{nI}} - \boldsymbol{p_{ne}} \tag{65}$$

where P is the pressure of the neutrals, Φ the gravitational potential and $\boldsymbol{p_{nI}}$ ($\boldsymbol{p_{ne}}$) is the momentum exchange rate between the neutrals and the ions (electrons).

The ion equation is

$$\rho_I\frac{\partial \boldsymbol{v_I}}{\partial t} + \rho_I(\boldsymbol{v_I}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v_I} = eZn_I(\boldsymbol{E} + \boldsymbol{v_I}\times\boldsymbol{B}) - \boldsymbol{\nabla}P_I - \rho_I\boldsymbol{\nabla}\Phi - \boldsymbol{p_{In}} \tag{66}$$

and the electron equation is

$$\rho_e\frac{\partial \boldsymbol{v_e}}{\partial t} + \rho_e(\boldsymbol{v_e}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v_e} = -en_e(\boldsymbol{E} + \boldsymbol{v_e}\times\boldsymbol{B}) - \boldsymbol{\nabla}P_e - \rho_e\boldsymbol{\nabla}\Phi - \boldsymbol{p_{en}} \tag{67}$$

The subscript I (e) refers to the ions (electrons). When not in a subscript but used in an equation, e is the fundamental charge of a proton, i.e. *it is always* *positive.* The electron charge is *always* $-e$. The momentum exchange rate $\boldsymbol{p_{In}}$ is precisely $-\boldsymbol{p_{nI}}$, and the same holds for $\boldsymbol{p_{en}}$. (Why?) The quantity Z is the mean charge per ion, n is a number density, and the fluid is neutral in bulk, $eZn_I = en_e$.

The key point is that for the charge carriers, all terms proportional to the mass densities ρ_I and ρ_e are small compared with the Lorentz force and momentum exchange rates. Hence, to a very good approximation,

$$0 = eZn_I(\boldsymbol{E} + \boldsymbol{v_I} \times \boldsymbol{B}) - \boldsymbol{p_{In}} \tag{68}$$

$$0 = -en_e(\boldsymbol{E} + \boldsymbol{v_e} \times \boldsymbol{B}) - \boldsymbol{p_{en}} \tag{69}$$

Adding these two equations and using bulk neutrality leads to ,

$$0 = eZn_I(\boldsymbol{v_I} - \boldsymbol{v_e}) \times \boldsymbol{B} - \boldsymbol{p_{In}} - \boldsymbol{p_{en}} \tag{70}$$

But $eZn_I(\boldsymbol{v_I} - \boldsymbol{v_e})$ is just the current density $\boldsymbol{J}$, so that

$$\boldsymbol{p_{In}} + \boldsymbol{p_{en}} = \boldsymbol{J} \times \boldsymbol{B} \tag{71}$$

Using this in the neutral equation leads to

$$\frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} P - \rho\boldsymbol{\nabla}\Phi + \boldsymbol{J} \times \boldsymbol{B} \tag{72}$$

Remarkably, the net Lorentz force appears unmodified in the equation for the *neutrals.*

For a sparsely ionized fluid, departures from ideal MHD appear mainly in the induction equation. To relate $\boldsymbol{E}$ and $\boldsymbol{B}$ it is best to use the electron force equation, since the ions may be more closely locked to the neutrals. Thus

$$\boldsymbol{E} = -\boldsymbol{v_e} \times \boldsymbol{B} - \frac{\boldsymbol{p_{en}}}{en_e} = -[\boldsymbol{v} + (\boldsymbol{v_e} - \boldsymbol{v_I}) + (\boldsymbol{v_I} - \boldsymbol{v})] \times \boldsymbol{B} - \frac{\boldsymbol{p_{en}}}{en_e} \tag{73}$$

Now matters start to get very detailed. I present these details in the following section, but for purposes of this course I view this material as entirely optional. Having made the details available to you, however, I feel free to make a quick summary of the results, leaving it for you to read the next section if you wish more explanation.

The term $\boldsymbol{v_e} - \boldsymbol{v_I}$ is $-\boldsymbol{J}/en_e$.

The term $\boldsymbol{v_I} - \boldsymbol{v}$ is related to $\boldsymbol{p_{In}}$ by an equation of the form

$$\boldsymbol{p_{In}} = \gamma\rho\rho_I(\boldsymbol{v_I} - \boldsymbol{v})$$

where γ is a coefficient that may be calculated from knowledge of the interaction cross sections. (See equation (83).) But $\boldsymbol{p_{In}}$ is simply related to $\boldsymbol{J} \times \boldsymbol{B}$ from equation (71), because $\boldsymbol{p_{In}}$ is in fact dominant over $\boldsymbol{p_{en}}$. Ultimately, the reason is that the ions are more massive than the electrons.

The final term proportional to $\boldsymbol{p_{en}}$ represents ohmic dissipation. We denote the electrical conductivity by σ_{cond}.

Putting all of this together leads to the full induction equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left[\boldsymbol{v} \times \boldsymbol{B} - \frac{(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}}{\mu_0 e n_e} + \frac{[(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}] \times \boldsymbol{B}}{\mu_0 \gamma \rho \rho_I} - \frac{\boldsymbol{\nabla} \times \boldsymbol{B}}{\mu_o \sigma_{cond}} \right] \tag{74}$$

The Details....

Let us examine matters a little more closely. (Please don't worry about every last detail. My purpose here is to give you a feeling for all that goes into a calculation like this, and to be able to understand the nomenclature that you will encounter in the literature. You don't have to become an expert in the minutiae of interstellar kinetic theory for this course!) $\boldsymbol{p_{nI}}$ takes the form

$$\boldsymbol{p_{nI}} = n \mu_{nI} (\boldsymbol{v} - \boldsymbol{v_I}) \nu_{nI} \tag{75}$$

where n is the number density of neutrals, and μ_{nI} is the reduced mass of an ion–neutral particle pair,

$$\mu_{nI} \equiv \frac{m_I m_n}{m_I + m_n}, \tag{76}$$

m_I and m_n being the ion and neutral mass respectively. ν_{nI} is the collision frequency of a neutral with a population of ions,

$$\nu_{nI} = n_I \langle \sigma_{nI} w_{nI} \rangle. \tag{77}$$

In equation (77), n_I is the number density of ions, σ_{nI} is the cross section for neutral-ion collisions, and w_{nI} is the relative velocity between a neutral particle and an ion. The angle brackets represent an average over all possible relative velocities in the thermal population of particles. Notice that equation (75) has the dimensions of a force per unit volume, and that it is proportional to the velocity difference between the species: if there is no difference in their mean velocities, two population of particles cannot exchange momentum.

Why does the reduced mass μ_{nI} appear? Because the reduced mass *always* appears in any interaction between two individual particles: in the center of mass frame the equations reduce to a single particle equation with the particle mass equal to the reduced mass. In an elastic one-dimensional collision, for example, if v is initial relative velocity of the two interacting particles, then the momentum exchange is $2\mu_{12}v$, where μ_{12} is the reduced mass. (Show this.)

For neutral-ion scattering, we may approximate the cross section σ_{nI} to be geometrical, which means that the quantity in angle brackets will be proportional to $\mu_{nI}^{-1/2}$. The order of the subscripts has no particular significance in either the cross section σ_{nI}, reduced mass μ_{nI}, or relative velocity w_{nI}. But ν_{In} does differ from ν_{nI}: the former is proportional to the neutral density n, the latter to the ion density n_I.

Putting all these definitions together gives

$$\boldsymbol{p_{nI}} = nn_I\mu_{nI}\langle\sigma_{nI}w_{nI}\rangle(\boldsymbol{v} - \boldsymbol{v_I}) \tag{78}$$

In accordance with Newton's third law, this is symmetric with respect to the interchange $n \leftrightarrow I$, except for a change in sign, $\boldsymbol{p_{nI}} = -\boldsymbol{p_{In}}$. All of these considerations hold, of course, for electron-neutral scattering as well. Explicitly, we have

$$\boldsymbol{p_{ne}} = nn_e\mu_{ne}\langle\sigma_{ne}w_{ne}\rangle(\boldsymbol{v} - \boldsymbol{v_e}) \simeq nn_em_e\langle\sigma_{ne}w_{ne}\rangle(\boldsymbol{v} - \boldsymbol{v_e}). \tag{79}$$

The gas is assumed to be locally neutral, so that $n_e = Zn_i$ where Z is the number of ionizations per ion particle. In a weakly ionized gas, $Z = 1$. The reduced mass μ_{ne} is very nearly equal to the electron mass m_e. The collision rates are given by (see Draine, Roberge, & Dalgarno 1983 ApJ 264, 485 for yet more details) (note, cgs units!):

$$\langle\sigma_{nI}w_{nI}\rangle = 1.9 \times 10^{-9} \text{ cm}^3 \text{ s}^{-1} \tag{80}$$

$$\langle\sigma_{ne}w_{ne}\rangle = 10^{-15} \ (128kT/9\pi m_e)^{1/2} = 8.3 \times 10^{-10}T^{1/2} \text{ cm}^3 \text{ s}^{-1} \tag{81}$$

The electron-neutral collision rate is just the ion geometric cross section times an electron thermal velocity. (The peculiar factor of $(128/9\pi)^{1/2}$ is a detail of the averaging procedure.) But the ion-neutral collision rate is temperature independent, much more beholden to long range induced dipole interactions, and significantly enhanced relative to a geometrical cross section assumption. Even if the ion-neutral rate were determined only by a geometrical cross section, $|\boldsymbol{p_{nI}}|$ would exceed $|\boldsymbol{p_{ne}}|$ by a factor of order $(m_e/\mu_{nI})^{1/2}$. In fact, the dipole enhancement of the ion-neutral cross section makes this factor larger still[2].

[2] I should be a little bit more careful. The statement that $|\boldsymbol{p_{ne}}|$ is larger than $|\boldsymbol{p_{nI}}|$ by a factor of $(m_e/\mu_{nI})^{1/2}$ assumes that the velocity differences $\boldsymbol{v} - \boldsymbol{v_e}$ and $\boldsymbol{v} - \boldsymbol{v_I}$ do not introduce any mass dependencies, which is generally true.

In the astrophysical literature, it is common to write the ion-neutral momentum coupling in the form

$$\boldsymbol{p_{In}} = \rho\rho_I\gamma(\boldsymbol{v_I} - \boldsymbol{v}), \tag{82}$$

where γ is the so-called *drag coefficient*,

$$\gamma \equiv \frac{\langle\sigma_{nI}w_{nI}\rangle}{m_I + m_n} \tag{83}$$

and we will use this notation from here on. Numerically, $\gamma = 3 \times 10^{13}$ cm^3 s^{-1} g^{-1} for astrophysical mixtures (Draine, Roberge, & Dalgarno 1983).

We come next to the ions and electrons. The dynamical equations for the ions and electrons are

$$\rho_I\frac{\partial\boldsymbol{v_I}}{\partial t} + \rho_I\boldsymbol{v_I}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v_I} = -\boldsymbol{\nabla}P_I - \rho_I\boldsymbol{\nabla}\Phi + Zen_I\left(\boldsymbol{E} + \boldsymbol{v_I}\boldsymbol{\times}\boldsymbol{B}\right) - \boldsymbol{p_{In}} \tag{84}$$

and

$$\rho_e\frac{\partial\boldsymbol{v_e}}{\partial t} + \rho_e\boldsymbol{v_e}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v_e} = -\boldsymbol{\nabla}P_e - \rho_e\boldsymbol{\nabla}\Phi - en_e\left(\boldsymbol{E} + \boldsymbol{v_e}\boldsymbol{\times}\boldsymbol{B}\right) - \boldsymbol{p_{en}}, \tag{85}$$

respectively. e will *always* denote the *positive* charge of a proton, the absolute value of the electron charge, 1.602×10^{-19} Coulombs or 4.803×10^{-10} esu.[3] For a weakly ionized gas, the Lorentz force and collisional terms dominate in each of the latter two equations. Comparison of the magnetic and inertial forces, for example, shows that the latter are smaller than the former by the ratio of the proton or electron gyroperiod to a macroscopic flow crossing time. Thus, to an excellent degree of approximation,

$$Zen_I\left(\boldsymbol{E} + \boldsymbol{v_I}\boldsymbol{\times}\boldsymbol{B}\right) - \boldsymbol{p_{In}} = 0, \tag{86}$$

and

$$-en_e\left(\boldsymbol{E} + \boldsymbol{v_e}\boldsymbol{\times}\boldsymbol{B}\right) - \boldsymbol{p_{en}} = 0. \tag{87}$$

The sum of these two equations gives

$$\boldsymbol{J}\boldsymbol{\times}\boldsymbol{B} = \boldsymbol{p_{In}} + \boldsymbol{p_{en}} \tag{88}$$

where charge neutrality $n_e = Zn_I$ has been used, and we have introduced the current density

$$\boldsymbol{J} \equiv en_e(\boldsymbol{v_I} - \boldsymbol{v_e}). \tag{89}$$

[3]Beware: esu units are still commonly used in the astrophysical literature! You should become comfortable with them.

The equation for the neutrals becomes

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi + \boldsymbol{J}\times\boldsymbol{B} \tag{90}$$

Due to collisional coupling, the neutrals are subject to the magnetic Lorentz force just as though they were a gas of charged particles. It is not the magnetic force *per se* that changes in a neutral gas. As well shall presently see, it is the inductive properties of the gas.

Let us return to the force balance equations for the electrons:

$$-en_e\left(\boldsymbol{E} + \boldsymbol{v_e}\times\boldsymbol{B}\right) - \boldsymbol{p_{en}} = 0. \tag{91}$$

After division by $-en_e$, this may be expanded to

$$\boldsymbol{E} + \left[\boldsymbol{v} + (\boldsymbol{v_e} - \boldsymbol{v_I}) + (\boldsymbol{v_I} - \boldsymbol{v})\right]\times\boldsymbol{B} + \frac{m_e\nu_{en}}{e}\left[(\boldsymbol{v_e} - \boldsymbol{v_I}) + (\boldsymbol{v_I} - \boldsymbol{v})\right] = 0, \tag{92}$$

where we have introduced the collision frequency of an electron in a population of neutrals:

$$\nu_{en} = n\langle\sigma_{ne}w_{ne}\rangle. \tag{93}$$

We have written the electron velocity $\boldsymbol{v_e}$ in terms of the dominant neutral velocity $\boldsymbol{v}$ and the key physical velocity differences of our problem. It has already been noted that in equation (88), $\boldsymbol{p_{en}}$ is small compared with $\boldsymbol{p_{In}}$, provided that the velocity difference $|\boldsymbol{v_e} - \boldsymbol{v}|$ is not much larger than $|\boldsymbol{v_I} - \boldsymbol{v}|$. As we argued earlier, the $\boldsymbol{p_{en}}$ term in equation (88) is small relative to $\boldsymbol{p_{In}}$:

$$\boldsymbol{J}\times\boldsymbol{B} \simeq \boldsymbol{p_{In}} = nn_I\mu nI(\boldsymbol{v_I} - \boldsymbol{v})\nu_{nI}. \tag{94}$$

It then follows that the final term in equation (92)

$$\frac{m_e\nu_{en}}{e}(\boldsymbol{v_I} - \boldsymbol{v}),$$

which is proportional to $\boldsymbol{J}\times\boldsymbol{B}$, becomes small compared with the third term

$$(\boldsymbol{v_e} - \boldsymbol{v_I})\times\boldsymbol{B},$$

which also proportional to $\boldsymbol{J}\times\boldsymbol{B}$, by a factor of order $(m_e/\mu_{In})^{1/2}$. These simplifications allow us to write the electron force balance equation as

$$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} - \frac{\boldsymbol{J}\times\boldsymbol{B}}{en_e} - \frac{\boldsymbol{J}}{\sigma_{cond}} + \frac{(\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}}{\gamma\rho\rho_I} = 0, \tag{95}$$

where the electrical conductivity has been defined as

$$\sigma_{cond} \equiv \frac{e^2 n_e}{m_e\nu_{en}} \tag{96}$$

The associated resistivity η is

$$\eta = \frac{1}{\mu_0 \sigma_{cond}}, \tag{97}$$

which has units of m^2 s^{-1}. Numerically (e.g. Blaes & Balbus 1994 ApJ, 421, 163; Balbus & Terquem 2001, ApJ, 552, 235):

$$\eta = 0.0234 \left(\frac{n}{n_e}\right) T^{1/2} \text{ m}^2 \text{ s}^{-1} \tag{98}$$

Equation (95) is a general form of Ohm's law for a moving, multiple fluid system.

Next, we make use of two of Maxwell's equations. The first is Faraday's induction law:

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{99}$$

We substitute $\boldsymbol{E}$ from equation (95) to obtain an equation for the self-induction of the magnetized fluid,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla\times\left[\boldsymbol{v}\times\boldsymbol{B} - \frac{\boldsymbol{J}\times\boldsymbol{B}}{e n_e} + \frac{(\boldsymbol{J}\times\boldsymbol{B})\times\boldsymbol{B}}{\gamma\rho\rho_I} - \frac{\boldsymbol{J}}{\sigma_{cond}}\right] \tag{100}$$

It remains to relate the current density $\boldsymbol{J}$ to the magnetic field $\boldsymbol{B}$. This is accomplished by the second Maxwell equation,

$$\mu_0 \boldsymbol{J} = \nabla\times\boldsymbol{B} + \frac{\partial \boldsymbol{E}}{\partial t} \tag{101}$$

The final term in the above is the displacement current, and it may be ignored. Indeed, since we have not, and will not, use the "Gauss's Law" equation

$$\nabla\cdot\boldsymbol{E} = (e/\epsilon_0)(Z n_I - n_e), \tag{102}$$

we *must not* include the displacement current. In Appendix B, we show that departures from charge neutrality in $\nabla\cdot\boldsymbol{E}$ and the displacement current are both small terms that contribute at the same order: v^2/c^2. These must both be self-consistently neglected in nonrelativisitc MHD. (The final Maxwell equation $\nabla\cdot\boldsymbol{B} = 0$ adds nothing new. It is automatically satisfied by equation (99), as long as the initial magnetic field satifies this divergence free condition.) These considerations imply

$$\boldsymbol{J} = \frac{1}{\mu_0}\nabla\times\boldsymbol{B} \tag{103}$$

for use in equation (100).

To summarize, the fundamental equations of a weakly ionized fluid are mass conservation of the dominant neutrals (eq.[64])

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{v}) = 0, \tag{104}$$

the equation of motion (eq. [90] with [103])

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}, \tag{105}$$

and the induction equation (eq. [100] with [97] and [103])

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times \left[\boldsymbol{v}\times\boldsymbol{B} - \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{\mu_0 e n_e} + \frac{[(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}]\times\boldsymbol{B}}{\mu_0 \gamma \rho \rho_I} - \frac{\boldsymbol{\nabla}\times\boldsymbol{B}}{\mu_o \sigma_{cond}} \right] \tag{106}$$

It is only natural that the reader should be a little taken aback by the sight of equation (106). Be assured that it is rarely, if ever, needed in full generality: almost always one or more terms on the right side of the equation may be discarded. When only the induction term $\boldsymbol{v}\times\boldsymbol{B}$ is important, we refer to this regime as *ideal* MHD. The three remaining terms on the right are the nonideal MHD terms.

To get a better feel for the relative importance of the nonideal MHD terms in equation (100), we denote the terms on the right side of the equation, moving left to right, as I (induction), H (Hall), A (ambipolar diffusion), and O (Ohmic resistivity). We will always be in a regime in which the presence of the induction term is not in question. More interesting is the relative importance of the nonideal terms. The explicit dependence of A/H and O/H in terms of the fluid properties of a cosmic gas has been worked out by Balbus & Terquem (2001):

$$\frac{A}{H} = Z \left(\frac{9\times 10^{12}\ \mathrm{cm}^{-3}}{n} \right)^{1/2} \left(\frac{T}{10^3\,\mathrm{K}} \right)^{1/2} \left(\frac{v_A}{c_S} \right) \tag{107}$$

and

$$\frac{O}{H} = \left(\frac{n}{8\times 10^{17}\ \mathrm{cm}^{-3}} \right)^{1/2} \left(\frac{c_S}{v_A} \right) \tag{108}$$

Here n is the total number density of all particles, T is the kinetic temperature, v_A is the so-called *Alfvén velocity* (much more about this quantity will come later!),

$$\boldsymbol{v_A} = \frac{\boldsymbol{B}}{\sqrt{\mu_0 \rho}} \tag{109}$$

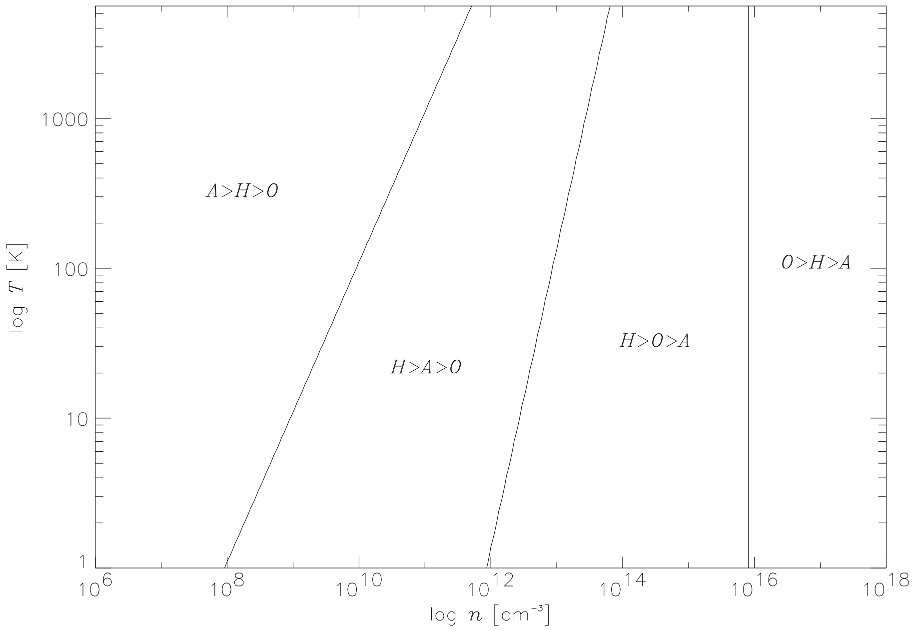

Figure 2: Parameter space for nonideal MHD. The curves correspond to the case $v_A/c_S = 0.1$. (From Kunz & Balbus 2004, MNRAS, 348, 355.)

and c_S is the isothermal speed of sound,

$$c_S^2 = 0.429 \frac{kT}{m_p} \tag{110}$$

where k is the Boltzmann constant and m_p the mass of the proton. The coefficient 0.429 corresponds to a mean mass per particle of $2.33 m_p$, appropriate to a molecular gas with a 10% helium admixture.

As reassurance that the fully general nonideal MHD induction equation is not needed for our purposes, note that equations (107) and (108) imply that for all three nonideal MHD terms to be comparable, $T \sim 10^8$ K! Obviously this is not a weakly ionized regime. In figure (2), we plot the domains of relative dominance of the nonideal MHD terms in the nT plane.

Our emphasis of the relative ordering of the nonideal terms in the induction equation should not obscure the fact that ideal MHD is often an excellent approximation, even when the ionization fraction is $\ll 1$. For example, the ratio of the ideal inductive term to the ohmic loss term is given by the Lundquist number

$$\boldsymbol{\ell} = \frac{v_A H}{\eta} \tag{111}$$

where H is a characteristic gradient length scale. To orient ourselves, let us consider the case of a protostellar disk and set $H = 0.1R$, where R is the radial location in the disk. (This would correspond to H being about the vertical thickness of the disk.) Then $\boldsymbol{\ell}$ is given by

$$\boldsymbol{\ell} \simeq 2.5(n_e/n)(v_A/c_S)R_{cm},$$

R_{cm} being the radius in centimeters. In other words, the critical ionization fraction at which $\boldsymbol{\ell} = 1$ is about

$$(n_e/n)_{crit} = 0.4(c_S/v_A)R_{cm}^{-1} \sim 10^{-13}(c_S/10v_A)$$

at $R = 1$ AU. The actual ionization fraction at this location may be above or below this during the course of the solar systems evolution, but the point worth noting here is that R_{cm} is a large number for a protostellar disk! Ionization fractions far, far below unity can render an astrophysical gas a near perfect electrical conductor. It therefore makes a great deal of sense to begin by examining the behavior of an ideal MHD fluid.

Exercise. Show that the Lorentz force may be written

$$\boldsymbol{J} \times \boldsymbol{B} = \partial_i \left(\frac{B_i B_j}{\mu} - \delta_{ij} \frac{B^2}{2\mu} \right) \equiv \partial_i T^L_{ij}. \tag{112}$$

Exercise. Show that the Newtonian self-gravity force may be written

$$-\rho \boldsymbol{\nabla} \Phi = \partial_i \left(-\frac{g_i g_j}{4G\pi} + \delta_{ij} \frac{g^2}{8G\pi} \right) \equiv \partial_i T^N_{ij}. \tag{113}$$

where $g_i = -\partial_i \Phi$. (Hint: $\partial_i \partial_i \Phi = 4\pi G\rho$.)

Exercise. Show that the inertial terms in the equation of motion be written

$$\rho \partial_t v_i + \rho v_j \partial_j v_i + \partial_i P = \partial_t(\rho v_i) + \partial_i \left(\rho v_i v_j + \delta_{ij} P \right) \equiv \partial_t(\rho v_i) + \partial_i T^R_{ij}, \tag{114}$$

which defines the Reynolds stress T^R_{ij}.

Exercise. Show that the equation of motion may be written

$$\rho \partial_t v_i + \partial_i T_{ij} = 0, \tag{115}$$

where $T_{ij} = T^L_{ij} + T^N_{ij} + T^R_{ij}$ is the energy-momentum stress tensor. This form of the equation of motion is most readily generalized when relativity becomes important.

4 Ideal MHD

4.1 Alfvén Waves

We begin our study with the behavior of an ideal, perfectly conducting, magnetized gas. Consider a constant density, constant pressure medium containing a uniform magnetic field. Define the z axis to lie along the direction of the magnetic lines of force. The x and y axes lies perpendicular to z, and time is denoted t. We wish to study the response of the gas when it is subject to very small perturbations which have the mathematical form $exp(ikz - i\omega t)$. The constant k is the wavenumber of the disturbance, and the constant ω is the angular frequency of the disturbance. As always, when we use the notation of a complex-valued exponential, we take the real part to obtain the physical quantity: $e^{i\alpha}$ corresponds to $\cos\alpha$, $ie^{i\alpha}$ corresponds to $-\sin\alpha$, and so forth.

To begin, we may write the induction in the form (just as was done for the vorticity equation):

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla}\times(\boldsymbol{B}\times\boldsymbol{v}) = \frac{D\boldsymbol{B}}{Dt} - (\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{B}\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} = 0 \tag{116}$$

Because the perturbations are very small, only terms that are linear in the amplitudes need be retained. Terms like $(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v}$, or example, may be dropped.

Let us begin by assuming that all three components of the velocity are present, as are density perturbations $\delta\rho$, pressure perturbations (δP), and magnetic field perturbations $\delta\boldsymbol{B}$. We will quickly find that many of these quantities vanish. For example, the divergences of the velocity $\boldsymbol{v}$ and the perturbed magnetic field $\delta\boldsymbol{B}$ both vanish. Since the pertubation depend only upon z, both δB_z and v_z vanish. The z equation of motion is

$$0 = -\frac{\partial}{\partial z}(\delta P + \delta\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{B}/\mu_0) \tag{117}$$

But $\delta\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{B}$ must vanish for an equilibrium field pointing in the z direction, so we conclude that δP vanishes. And, for adiabatic perturbations, $\delta\rho$ must likewise vanish. For the remaining x and y velocity components, the equation of motion is

$$\frac{\partial \boldsymbol{v}}{\partial t} = \frac{B}{\mu_0\rho}\frac{\partial \delta\boldsymbol{B}}{\partial z} \tag{118}$$

or

$$-i\omega\boldsymbol{v} = (ikB\mu_0)\delta\boldsymbol{B} \tag{119}$$

The vectors $\boldsymbol{v}$ and $\delta\boldsymbol{B}$ are parallel, and without loss of generality, we will consider linearly polarized waves pointing along the x axis. (The waves may be in principle circularly polarized, but their propagation properties are the same.) The induction equation gives

$$-i\omega\delta\boldsymbol{B} = ikB\boldsymbol{v} \tag{120}$$

This is consistent with the equation of motion if and only if

$$\omega^2 = k^2B^2/(\rho\mu_0) \tag{121}$$

We define the Alfvén velocity as

$$\boldsymbol{v_A} = \frac{\boldsymbol{B}}{\sqrt{\rho\mu_0}} \tag{122}$$

so that our wave dispersion relation

$$\omega^2 = k^2v_A^2 \tag{123}$$

These waves propagate along the magnetic lines of force with a velocity v_A, and induce zero pressure and density changes. They are *transverse* waves. Alvén waves are thought to play an important role in the heating of the solar corona (where they are damped), and may be important in the interstellar medium for heating interstellar clouds–in the case to relatively cool temperatures.

4.2 Fast, slow, and Alfvén waves

Let us consider more general disturbances in a magnetized medium. We consider an equilibrium field with components in the z and y direction. The wave number $\mathbf{k}$ lies along the z axis. The perturbed magnetic field in the z direction, δB_z must therefore vanish. With small perturbations denoted by a leading δ, the governing linear equations are mass conservation,

$$-\omega\frac{\delta\rho}{\rho} + k\delta v_z = 0 \tag{124}$$

the equations of motion,

$$-i\omega\,\delta v_x - i\frac{kB_z}{\mu_0\rho}\delta B_x = 0, \tag{125}$$

$$-i\omega\,\delta v_y - i\frac{kB_z}{\mu_0\rho}\delta B_y = 0, \tag{126}$$

$$-i\omega\,\delta v_z + ik\left(\frac{\delta P}{\rho} + \frac{B_y \delta B_y}{\mu_0 \rho}\right) = 0, \tag{127}$$

and the induction equation,

$$-i\omega \delta B_x = ikB_z \delta v_x, \tag{128}$$

$$-i\omega \delta B_y = ikB_z \delta v_y. - B_y ik \delta v_z \tag{129}$$

Finally, the entropy equation for adiabatic disturbances is simply

$$\frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho}. \tag{130}$$

It is a straightforward, if somewhat lengthy, exercise to solve for the dispersion relation:

$$[\omega^2 - (\boldsymbol{k \cdot v_A})^2][\omega^4 - k^2\omega^2(a^2 + v_A^2) + (\boldsymbol{k \cdot v_A})^2 k^2 a^2] = 0. \tag{131}$$

Here, $a^2 = \gamma P/\rho$ is the square of the isothermal sound speed.

The Alfvén branch of the dispersion relation is explicitly decoupled from the others. Let us write their frequency as ω_A, and let the angle between the wavevector $\boldsymbol{k}$ and the equilibrium $\boldsymbol{B}$ be θ. Then $\omega_A = kv_A \cos\theta$.

The remaining roots (of the quartic polynomial) of our dispersion relation correpond to what are known as the "fast mode" and the "slow mode." These are most easily understood when either i) $a \ll v_A$; ii) $v_A \ll a$; iii) $\cos\theta \ll 1$. Then, one solution to the quartic, the fast mode, is a balance between the first two terms,

$$\omega_+^2 = k^2(v_A^2 + a^2) \tag{132}$$

and the other solution, the slow mode, is a balance between the last two terms,

$$\omega_-^2 = \frac{k^2 v_A^2 a^2 \cos^2\theta}{v_A^2 + a^2} \tag{133}$$

Notice that $\omega_+ > \omega_A > \omega_-$.

The ω_+ modes comes from magnetic and pressure forces acting together. It is sometimes referred to as a *magnetosonic wave.* The ω_- mode corresponds to pressure and magnetic forces in opposition. When the magnetic field is strong, the result is an ordinary sound wave channeled along the field lines. When the field is weak, the slow mode becomes degenerate with an Alfvén wave.

The slow mode can also have a different frequency from an Alfvén wave when the medium is in rotation, even in the limit of very large a. In fact, it is possible for ω_-^2 to pass through zero and become negative! This corresponds to imaginary ω and an exponenitally growing mode. This is the magnetorotational instability, or MRI, a very important instability for understanding the origin of turbulence accretion disks. We will have much to say about the MRI later in the course.

4.3 Small Perturbations: a More Formal Treatment

4.3.1 Linear, nonlinear, Eulerian, Lagrangian

Waves are said to be *linear* if their associated amplitudes are much smaller than the corresponding equilibrium values of the background medium. Otherwise they are *nonlinear.* For example, if at a particular point in a fluid the equilibrium pressure is $P(\boldsymbol{r})$, and a wave disturbance at time t causes the pressure to change to $P'(\boldsymbol{r}, t)$, then in linear theory,

$$P'(\boldsymbol{r}, t) - P(\boldsymbol{r}) \equiv \delta P \ll P(\boldsymbol{r}) \tag{134}$$

For the velocity, linear theory generally requires the disturbance to be much less than $\sqrt{P/\rho}$, *not* the velocity of the background. The flow velocity itself is irrelevant, since relative motion by itself does not affect local physics! (Velocity *gradients* in the equilibrium flow are a different matter, however. They can, in fact, be critical for understanding wave propagation.) The name "linear" refers to the fact that in the mathematical analysis, only terms *linear* in the δ amplitudes are retained, while terms of quadratic or higher order are ignored.

Small disturbances can be described mathematically in more than one way. The above equation for δP is known as an Eulerian perturbation, which is the difference between the equilibrium and perturbed values of a fluid quantity taken *at a fixed point in space.* It is sometimes useful to work with what is known as a Langrangian perturbation, particularly when freely moving boundary surfaces are present. In a Lagrangian disturbance, we focus not upon the change at a fixed location $\boldsymbol{r}$, but upon the changes associated with a particular *fluid element* when it undergoes a displacement $\boldsymbol{\xi}$. For the case of a pressure disturbance, for example, we ask ourselves how does the pressure of a fluid element change when it is displaced from its equilibrium value $\boldsymbol{r}$ to $\boldsymbol{r} + \boldsymbol{\xi}$? The Langrangian perturbation ΔP is therefore

$$P'(\boldsymbol{r} + \boldsymbol{\xi}, t) - P(\boldsymbol{r}) \equiv \Delta P. \tag{135}$$

Note the difference between equations (134) and (135). To linear order in ξ, ΔP and δP are related by

$$\Delta P = P'(\boldsymbol{r}, t) - P(\boldsymbol{r}) + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} P = \delta P + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} P. \tag{136}$$

The Lagrangian velocity perturbation $\Delta \boldsymbol{v}$ is given by $D\boldsymbol{\xi}/Dt$:

$$\Delta \boldsymbol{v} \equiv \frac{D\boldsymbol{\xi}}{Dt} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} \tag{137}$$

where $\boldsymbol{v}$ is any background velocity that is present. This is simply the instantaneous time rate of change of the displacement of a fluid element, taken relative to the unperturbed flow. Since

$$\Delta \boldsymbol{v} = \delta \boldsymbol{v} + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{v}, \tag{138}$$

the Eulerian velocity perturbation $\delta \boldsymbol{v}$ is related to the fluid displacement $\boldsymbol{\xi}$ by:

$$\delta \boldsymbol{v} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \tag{139}$$

Exercise. Let $\boldsymbol{v} = R\Omega(R)\boldsymbol{e_\phi}$. Consider a displacement $\boldsymbol{\xi}$ with radial and azimuthal components ξ_R and ξ_ϕ, each depending upon R and ϕ. Show that

$$\frac{D\xi_R}{Dt} = \delta v_R, \quad \frac{D\xi_\phi}{Dt} = \delta v_\phi + \xi_R \frac{d\Omega}{d\ln R}$$

where $D/Dt = \partial/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla}$. (Be careful!)

4.3.2 Equations of Constraint

Imagine an equilibrium background, independent of time. A finite velocity disturbance is made over a tiny period of time that causes the flow to change. The partial derivative $\partial/\partial t$ come to life.

Notice that a term like $\partial\rho/\partial t$ has an interesting interpretation in this case: it is $\delta\rho/\delta t$, where $\delta\rho$ is an Eulerian perturbation and δt is just a time interval. The perturbation is a true Eulerian perturbation because it is the change in ρ at constant $\boldsymbol{r}$. Thus, mass conservation may be written

$$\delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\delta t) \tag{140}$$

The velocity is the unperturbed velocity $\boldsymbol{v_0}$ (which may be zero) plus the disturbed velocity $\boldsymbol{v_1}$ which here is comparable to $\boldsymbol{v_0}$ (or infinitely larger if the latter is zero). Since $\boldsymbol{v_1}\delta t$ is $\boldsymbol{\xi}$, and the $\boldsymbol{v_0}$ term causes no time change by definition, we obtain

$$\delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}) \tag{141}$$

This may also be written

$$\Delta\rho = -\boldsymbol{\nabla}\cdot\boldsymbol{\xi}. \tag{142}$$

We have, in effect, integrated the mass conservation equation, an equation of constraint in the sense that it imposes a restriction on the behavior of the flow: only displacements satisfying either of the above two (equivalent) equations are allowed, regardless of whether the disturbances are *dynamically* acceptable or not!

Exactly similar reasoning for adiabatic perturbations leads to

$$\delta S = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla} S \tag{143}$$

for the entropy S, and to

$$\delta \boldsymbol{B} = \boldsymbol{\nabla} \times (\boldsymbol{\xi} \times \boldsymbol{B}) \tag{144}$$

for the magnetic field. The usefulness of these equations will shortly become apparent.

4.4 The Virial Theorem

4.4.1 Analysis

The Virial Theorem is one of the most useful theorems in astrophysical gas-dynamics. Basically, it is an integrated form of the equation of motion in full generality. When the dominant balance is between two forces, the theorem states that the associated energies must be comparable in strength. We shall use Cartesian index notation in our proof.

Begin with

$$\rho \frac{Dv_i}{Dt} = -\partial_i P - \partial_i \left(\frac{B^2}{2\mu} \right) - \rho \partial_i \Phi + \frac{B_j}{\mu} \partial_j B_i \tag{145}$$

where

$$\Phi(\boldsymbol{r}) = -G \int \frac{\rho(\boldsymbol{r'})\, d^3 r'}{|\boldsymbol{r} - \boldsymbol{r'}|} \tag{146}$$

is the gravitational potential the system. Note that

$$-\partial_i \Phi = -G \int \frac{\rho(\boldsymbol{r'})\,(r_i - r_i')\, d^3 r'}{|\boldsymbol{r} - \boldsymbol{r'}|^3} \tag{147}$$

Multiply the equation of motion by r_i and sum over i,

$$\rho r_i \frac{Dv_i}{Dt} = -r_i \partial_i P - r_i \partial_i \left(\frac{B^2}{2\mu} \right) - \rho r_i \partial_i \Phi + r_i \frac{B_j}{\mu} \partial_j B_i \tag{148}$$

and then integrate over a fixed volume V. For the pressure integral,

$$\begin{aligned} -\int r_i\partial_i P\,dV &= -\int \partial_i(r_iP)\,dV + 3\int P\,dV && (149)\\ &= -\int P\boldsymbol{r}\cdot\boldsymbol{dA} + 3\int P\,dV \\ &= -\int P\boldsymbol{r}\cdot\boldsymbol{dA} + 2\int U_{therm}\,dV && (150)\end{aligned}$$

where $U_{therm} = (3/2)P$ is the thermal energy density.

The integral involving the potential is

$$\int \rho r_i \frac{\partial\Phi}{\partial r_i}\,d^3r = G\int \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r'})r_i(r_i - r_i')}{|\boldsymbol{r}-\boldsymbol{r'}|^3}\,d^3r\,d^3r' \tag{151}$$

If we switch the labels $\boldsymbol{r}$ and $\boldsymbol{r'}$, we obtain

$$\int \rho r_i \frac{\partial\Phi}{\partial r_i}\,d^3r = G\int \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r'})r_i'(r_i' - r_i)}{|\boldsymbol{r}-\boldsymbol{r'}|^3}\,d^3r\,d^3r' \tag{152}$$

Adding and dividing by 2:

$$\int \rho r_i \frac{\partial\Phi}{\partial r_i}\,d^3r = \frac{G}{2}\int \frac{\rho(\boldsymbol{r})\rho(\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|}d^3r\,d^3r' \equiv -V. \tag{153}$$

i.e., this is just minus the gravitational potential energy V. (The factor of 1/2 is present because each pair of interacting fluid elements occurs twice in the integration, but should only be counted once.)

On to the magnetic integrals:

$$\int r_i\partial_i(B^2/2\mu)\,d^3r = \int (B^2/2\mu)\boldsymbol{r}\cdot\boldsymbol{dA} - 3\int (B^2/2\mu)\,d^3r, \tag{154}$$

where we have integrated by parts and used the divergence theorem. And

$$\begin{aligned} \int \frac{r_i}{\mu}\partial_j(B_iB_j)d^3r &= \int (\boldsymbol{r}\cdot\boldsymbol{B})\frac{\boldsymbol{B}}{\mu}\cdot\boldsymbol{dA} - \int \frac{\delta_{ij}}{\mu}B_iB_j\,d^3r \\ &= \int (\boldsymbol{r}\cdot\boldsymbol{B})\frac{\boldsymbol{B}}{\mu}\cdot\boldsymbol{dA} - \int \frac{B^2}{\mu}\,d^3r. && (155)\end{aligned}$$

The sum of all the terms on the right side of our equation is then

$$2E_{therm} + V + M - \int\left(P + \frac{B^2}{2\mu}\right)\boldsymbol{r}\cdot\boldsymbol{dA} + \frac{1}{\mu}\int(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{B}\cdot\boldsymbol{dA} \tag{156}$$

where

$$E_{therm} = \int U_{therm}\, d^3r, \quad M = \int \frac{B^2}{2\mu}\, d^3r \tag{157}$$

are the total thermal and magnetic energies.

For the left side of the virial equation we start with the following identity:

$$\int \rho \frac{DQ}{Dt} d^3r = \int \rho \left(\frac{\partial Q}{\partial t} + \boldsymbol{v} \boldsymbol{\cdot} \boldsymbol{\nabla} Q \right) d^3r = \int \frac{\partial(\rho Q)}{\partial t}\, d^3r + \int \boldsymbol{\nabla} \boldsymbol{\cdot} (\rho \boldsymbol{v} Q)\, d^3r \tag{158}$$

where the second equality follows from mass conservation. The last integral can be converted to a surface integral of the flux $\rho \boldsymbol{v} Q$ over a bounding area. If we choose the surface so that the velocity vanishes at this surface, then this integral vanishes, and we shall make this assumption. We then have:

$$\int \rho \frac{DQ}{Dt}\, d^3r = \int \frac{\partial(\rho Q)}{\partial t}\, d^3r = \frac{d}{dt} \int \rho Q\, d^3r \tag{159}$$

With this in hand, we perform the following manipulations:

$$\begin{aligned} \int \rho r_i \frac{Dv_i}{Dt}\, d^3r &= \int \rho \frac{D(r_i v_i)}{Dt}\, d^3r - \int \rho v_i \frac{Dr_i}{Dt}\, d^3r && (160) \\ &= \frac{d}{dt} \int \rho r_i v_i\, d^3r - \int \rho v^2\, d^3r \\ &= \frac{d}{dt} \int \rho r_i \frac{Dr_i}{Dt}\, d^3r - 2KE \\ &= \frac{d}{dt} \int \frac{\rho}{2} \frac{Dr^2}{Dt}\, d^3r - 2KE \\ &= \frac{d^2}{dt^2} \frac{1}{2} \int \rho r^2 d^3r - 2KE \\ &= \frac{1}{2} \frac{d^2 I}{dt^2} - 2KE && (161) \end{aligned}$$

where I is $\int \rho r^2 d^3r$ and KE denotes the total kinetic energy of the fluid. The Virial theorem is then:

$$\frac{1}{2}\ddot{I} = 2KE + 2E_{therm} + V + M - \int \left(P + \frac{B^2}{2\mu} \right) \boldsymbol{r} \cdot \boldsymbol{dA} + \frac{1}{\mu} \int (\boldsymbol{r} \cdot \boldsymbol{B}) \boldsymbol{B} \cdot \boldsymbol{dA} \tag{162}$$

where the velocity is assumed to vanish over the bounding surface. The Virial theorem shows that when a dominant steady-state balance is present between two effects–pressure and gravity, say—the two associated energies are comparable. In particular, for a star in hydrostatic equilibrium,

$$E_{therm} = -\frac{1}{2}V, \quad E_{total} = E_{therm} + V = \frac{V}{2} \tag{163}$$

since the pressure vanishes at the surface, and magnetic fields are generally negligible for stellar hydrostatic equilibrium.

4.4.2 Interstellar Clouds

A classical application of the virial theorem is to interstellar clouds. Consider first a nonmagnetized spherical cloud that is embedded in an ambient pressure P. For steady conditions, the virial theorem states

$$2E_{therm} + V = \int P\boldsymbol{r} \cdot \boldsymbol{dA} \tag{164}$$

The thermal energy E_{therm} will be $(3/2)NkT$, where T is the average cloud temperature and N is the total number of particles. For N, we will write $N = M/\mu$, where M is the mass of the cloud and μ is the mass per particle. The potential energy V will be approximated by $-(3/5)GM^2/R$, the value for a constant density sphere. The surface pressure integral is $4\pi R^3 P_0$, where P_0 is the confining pressure. Hence,

$$P_0 = \frac{3MkT}{\mu 4\pi R^3} - \frac{3}{5}\frac{GM^2}{4\pi R^4} \tag{165}$$

Hold T and M fixed, and imagine a series of different equilibrium solutions in which R changes. When R is large, P_0 is small and increases as R gets smaller because the gravity term is less important. We can always find an R for a given P_0. (This is like squeezing a balloon.) Then gravity becomes more important, the pressure rises less rapidly as R decreases, and at some point if we squeeze more, there is no equilibrium R! This happens when dP_0/dR is zero. The radius at which this occurs is

$$R_m = \frac{4}{15}\frac{GM\mu}{kT} \tag{166}$$

and the resulting pressure is

$$P_m = \frac{5^3 3^4}{4^5 \pi}\left(\frac{kT}{\mu}\right)^4 \frac{1}{G^3 M^2} \tag{167}$$

The numerical coefficient is 3.15.

What is the effect of a magnetic field? As a very simple model of the field, we follow Spitzer's calculation and take the magnetic field to be uniform inside the cloud, and that the field strength $B^2(r)$ outside the cloud equals $B^2(R/r)^6$.

We now take the surface terms not at the cloud radius R, but at infinity. Then the surface integrals of the magnetic field may be ignored. The contribution to the magnetic energy density within the cloud is

$$(1/2\mu_0)\int_0^R B^2 4\pi r^2 dr = \frac{2\pi}{3\mu_0}B^2R^3 \tag{168}$$

and the contribution from outside the cloud is

$$(1/2\mu_0)\int_R^\infty R^6 B^2(4\pi/r^4)dr = \frac{2\pi}{3\mu_0}B^2R^3 \tag{169}$$

for a total of $(4\pi/3\mu_0)B^2R^3$.

What about the pressure terms, which appear to change when we move the surface to infinity? There is in fact no change from putting the surface at the cloud radius R! The increase in the thermal energy term is

$$3\int_R^\infty P_0 4\pi r^2 dr \tag{170}$$

and the surface term is now

$$-\int P_0 r^3\, d\Omega, \tag{171}$$

the integral being over solid angles at infinity. The sum of these two terms involves a cancellation of the diverging terms at infinity ($4\pi r^3 P_0$ for large r), and the contribution of the thermal energy integral and the lower limit is $-4\pi R^3 P_0$, which is just what we found before (but from the surface integral!).

The virial theorem now reads

$$P_0 = \frac{3MkT}{\mu 4\pi R^3} - \frac{3}{5}\frac{GM^2}{4\pi R^4} + \frac{B^2}{3\mu_0} \tag{172}$$

or

$$P_0 = \frac{3MkT}{\mu 4\pi R^3} - \frac{3}{5}\frac{GM^2}{4\pi R^4}\left[1 - \frac{20\pi B^2R^4}{9\mu_0 GM^2}\right] \tag{173}$$

The quantity in square brackets on the right is constant since the magnetic flux, πBR^2 is conserved. Hence, the analysis proceeds just as in the case of no magnetic field, but with a modified value of G: the old G multiplied by the square bracket term. Let us write this term as a function of a critical mass M_c, defined by equal magnetic and gravitational contributions. With $BR^2 = B_0 R_0^2$ (initial values), the mass M_c must satisfy

$$M_c^2 = \frac{20\pi B_0^2 R_0^4}{9\mu_0 G} = \frac{20\pi B_0^2}{9\mu_0 G}\left(\frac{3M_c}{4\pi\rho_0}\right)^{4/3} \tag{174}$$

or

$$M_c^{2/3} = \frac{20\pi B_0^2}{9\mu_0 G}\left(\frac{3}{4\pi\rho_0}\right)^{4/3} \tag{175}$$

For a cloud of a given initial radius $R_0 = (3M/4\pi\rho_0)^{1/3}$, the critical mass M_c is then a measure of the initial magnetic field strength B_0. The equation for P_0 is then

$$P_0 = \frac{3MkT}{\mu 4\pi R^3} - \frac{3}{5}\frac{GM^2}{4\pi R^4}\left[1 - \left(\frac{M_c}{M}\right)^{2/3}\right] \tag{176}$$

The equations for R_m and P_m that we first derived for a nonmagnetized cloud become

$$R_m = \frac{4}{15}\frac{GM\mu}{kT}\left[1 - \left(\frac{M_c}{M}\right)^{2/3}\right] \tag{177}$$

and

$$P_m = 3.15\left(\frac{kT}{\mu}\right)^4\left(\frac{1}{G^3 M^2}\right)\left[1 - \left(\frac{M_c}{M}\right)^{2/3}\right]^{-3} \tag{178}$$

5 The Newcomb-Parker Problem

Often just called the “Parker Instability” in the astrophysical literature, the more properly called “Newcomb-Parker Problem” addresses the behavior of a gas in a gravitational field with partial magnetic support. The problem had been formally formerly well-studied in the plasma physics community before astrophysicists took it up. If you’re going to try to magnetically confine thermonuclear reactions, you’d better know what you’re doing, so it is not surprising that the plasma physicists had thought about this long and hard.

The problem itself is as follows. If a disturbance is made to a gas that keeps everything in pressure equilibrium, an upward moving parcel cools. The equilibrium surrounding gas also typically become cooler as one moves up, but the fall in temperature is not as drastic as in the rising parcel. The parcel is therefore cooler and heavier than its surroundings, and it drops back down. Exactly the same reasoning works in reverse for a downward moving parcel, which returns back up. The restoring "buoyancy force," as it is called, allows a sort of wave to propagate in the gas, often referred to as a *gravity wave.*

When a magnetic field is present, the rising parcel in the above argument can have a lower density and still maintain pressure balance with the surroundings, thanks to the magnetic pressure contribution. Indeed, if the field is strong enough, the rising magnetized parcel could in fact be less dense than the surroundings, in which case the resulting buoyancy force would actually be *upwards.* The parcel is then accelerated in the same direction as its displacement, and the system is disrupted. An *instability* is said to be present.

Discussion of the Parker Instability in the astrophysical literature tends to be confused and hence unclear (i.e. wrong); in the plasma community the discussion is often too terse and mathematical. Here, I hope to find the right middle ground.

Consider a slab of gas lying in the xy plane with vertical coordinate z. There is a gravitational field $\boldsymbol{g} = -g\boldsymbol{e_z}$ pointing downward in the $-z$ direction. The gas contains a magnetic field, which in the equilibrium configuration lies in the x direction. The magnetohydrostatic equilibrium is given by

$$\boldsymbol{e_z}\frac{d}{dz}\left(P + \frac{B^2}{2\mu_0}\right) = \rho\boldsymbol{g} \tag{179}$$

As usual, P and ρ denote the gas pressure and density, respectively. All quantities may depend upon z.

We next consider perturbations corresponding to vertical motions. Since there is no x and t dependence in the equilibrium state, we are free to assume an xt dependence of $\exp(ikx - i\omega t)$ in the perturbations, where k is a constant wavenumber and ω the associated angular frequency. We will consider displacements in the vertical direction, $\boldsymbol{\delta v} = \delta v\boldsymbol{e_z}$.

In the analysis to follow, we will often use the displacement $\boldsymbol{\xi}$ instead of $\boldsymbol{\delta v}$, since the equations take a simpler form. For a problem in which the background equilibrium is static, the relation between the displacement amplitude ξ and δv is simply $\delta v = -i\omega\xi$. Finally, we will adopt the notation $\partial/\partial z = \partial_z$, etc., and work in the limit $\omega \to 0$, since we are interested in stability.

The linearly perturbed Eulerian dynamical equation of motion is

$$0 = -\frac{1}{\rho}\boldsymbol{\nabla}\left(\delta P + \frac{\boldsymbol{B}\cdot\boldsymbol{\delta B}}{\mu_0}\right) + \frac{\delta\rho}{\rho}\boldsymbol{g} + \frac{B\partial_x\boldsymbol{\delta B}}{\rho\mu_0} + \boldsymbol{e_x}\frac{\boldsymbol{\delta B}\cdot\boldsymbol{\nabla}B}{\rho\mu_0}, \tag{180}$$

while the induction equation for the magnetic field is

$$\boldsymbol{\delta B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B}) = \boldsymbol{\nabla}\times(\xi B\boldsymbol{e_y}) \tag{181}$$

or

$$\delta B_z = B\partial_x\xi, \qquad \delta B_x = -\xi\partial_z B. \tag{182}$$

Notice that the final term in equation (181) is not to be regarded as small!

There is a neat equality between certain magnetic force terms, since B does not depend on x:

$$\delta B_z \partial_z B = (B\partial_x\xi)\partial_z B = -B\partial_x \delta B_x. \tag{183}$$

This is a very handy result, for in view of (183), the final two terms in the x component of the equation of motion immediately cancel, leaving just

$$0 = -\partial_x \left(\delta P + \frac{\delta B_x\, B}{\mu_0}\right). \tag{184}$$

Since $\partial/\partial x$ amounts to multiplication by ik, we find the very reasonable result that if there are no horizontal accelerations, the perturbed gas and magnetic pressures must cancel one another:

$$\delta P = -\frac{\delta B_x\, B}{\mu_0} = \frac{B}{\mu_0}(\xi\partial_z B) = \xi\frac{\partial}{\partial z}\left(\frac{B^2}{2\mu_0}\right) \tag{185}$$

This, in turn means that the pressure terms make *no contribution* to the z equation of motion, which is now very simple indeed:

$$0 = g\frac{\delta\rho}{\rho} + (\boldsymbol{k}\cdot\boldsymbol{v_A})^2\xi \tag{186}$$

There are only two forces acting on a parcel of gas: buoyancy, and magnetic tension. It remains only to compute $\delta\rho/\rho$ for adiabatic disturbances,

$$\frac{\delta\rho}{\rho} = \frac{1}{\gamma}\frac{\delta P}{P} + \frac{\xi}{\gamma}\frac{\partial \ln P\rho^{-\gamma}}{\partial z} = \frac{\xi}{\gamma}\left(\frac{1}{P}\frac{\partial}{\partial z}\frac{B^2}{2\mu_0} + \frac{\partial \ln P\rho^{-\gamma}}{\partial z}\right) \tag{187}$$

An interesting way to write this result is to make use of the equilibrium condition (179)

$$\frac{\delta\rho}{\rho} = -\frac{\xi}{\gamma}\left(\frac{\rho g}{P} + \frac{\partial \ln \rho^{\gamma}}{\partial z},\right) \tag{188}$$

which altogether eliminates the explicit appearance of the magnetic field. Using this result in equation (186) yields the criterion for marginal stability:

$$0 = -g\left(\frac{g}{a^2} + \frac{\partial \ln \rho}{\partial z}\right) + (\boldsymbol{k}\cdot\boldsymbol{v_A})^2, \tag{189}$$

where $a^2 = \gamma P/\rho$ is, we shall later see, the classical expression for the speed of sound. The point is that instability is possible. The physical sense of the instability is that there must be sufficiently high density material near the midplane to anchor the magnetic field. If, on the other hand,

$$\frac{\partial \ln \rho}{\partial z} > -\frac{g}{a^2} \tag{190}$$

there is an instability: a small vertical disturbance is exponentially magnified. This is called a convective instability, since it leads to bulk convective motions.

The magnetic field in itself does not cause the convection to occur: with or without a magnetic field, convective instability is possible. Indeed, without a field, the criterion for instability is the same as saying that the entropy decreases upwards. This was found by Karl Scwarzschild in his classic study of convective instability in stars. The point is that with a magnetic field it is easier to generate convective instability, since the (negative) equilibrium density gradient can be larger (i.e. *smaller* in magnitude, closer to zero), thanks to the additional support of a magnetic field. This makes it easier to trigger convective, or buoyant, instability. The role of the magnetic field in destabilizing astrophysical gases was championed by Parker in 1966; plasma applications were first investigated years earlier (1961) by Newcomb. In astrophysics, this form of magnetic destabilization has come to be known as the *Parker Instability.*

Much has been made of the Parker process for forming stars in the spiral arms of galaxies (dense matter falls and collects into magnetic "wells"), but little is known with any degree of certainty. (My own view is that matters are much more complex in real galaxies, with vertical convection an ongoing process.) The student would be well-served to understand that many plausible and well-developed competing models of star formation abound in the literature.

Exercise. In Parker's treatment, the effects of cosmic rays are included! What does this mean? Cosmic rays are a population of relativistic particles with their own pressure, let us say P_{cr}. P_{cr} figures in the hydrostatic equilibrium:

$$\frac{dP_{tot}}{dz} = -\rho g, \quad P_{tot} = P + B^2/2\mu_0 + P_{cr}$$

The defining physical property of P_{cr} is that the pressure remains constant along a magnetic field line. This is because the cosmic rays can freely stream along the field lines, and they will not allow pressure deviations to be maintained along the way. Show that, for the problem of interest, the condition $\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla} P_{cr} = 0$ in linearized form is:

$$\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}\Delta P_{cr} = 0.$$

Hence, the Lagrangian change ΔP_{cr} does not change along a field line. In particular, if it is zero at one point along a field line, it is zero everywhere along the field line. (In our problem, $\boldsymbol{B}\cdot\boldsymbol{\nabla}$ amounts to multiplication by $\boldsymbol{k}\cdot\boldsymbol{B}$.)

With $\Delta P_{cr} = 0$, repeat the Parker problem just completed, but include cosmic rays. Show that the condition for instability (190) remains *unchanged.*

6 Accretion and Winds

6.1 Spherical Accretion: The Bondi Problem

Consider an extended gaseous medium with a central gravitating point mass M. The pressure and density in the medium are P_∞ and ρ_∞ respectively. The gravitational field of the point mass draws on the surroundings, and gas either accumulates onto the surface of the star, or is lost through the horizon of a black hole. This process is referred to as *accretion.* Accretion heats the gas by compression, and if there is turbulence, by dissipation as well. Gravitational accretion is thought to be the power source responsible for active galactic nuclei, the most luminous objects known.

6.1.1 Formulation

In its simplest form, accretion flow is spherical and time-steady. Under these conditions, the mass accretion rate

$$\dot{m} = -4\pi\rho r^2 v \tag{191}$$

is constant. Here, v is the radial velocity, and the minus sign is inserted so that $\dot{m} > 0$. The gas is assumed to follow a simple polytropic law

$$P = K\rho^\gamma \tag{192}$$

where the constant K may be defined by the gas properties at $r = \infty$:

$$K = P_\infty \rho_\infty^{-\gamma} \tag{193}$$

The speed of sound, an important parameter in this problem, is

$$a^2 = \frac{dP}{d\rho} = \gamma K \rho^{\gamma-1} \tag{194}$$

Thus, another useful way to write the constant K is

$$\gamma K = a_\infty^2 \rho_\infty^{1-\gamma}, \tag{195}$$

or more simply,

$$a^2 = a_\infty^2 \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} \tag{196}$$

The equation of motion is

$$v\frac{dv}{dr} = -\frac{1}{\rho}\frac{dP}{dr} - \frac{GM}{r^2} \tag{197}$$

But,

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{d}{dr}\left(\frac{\gamma K \rho^{\gamma-1}}{\gamma - 1}\right) = \frac{d}{dr}\left(\frac{a^2}{\gamma-1}\right), \tag{198}$$

which allows the equation of motion to be integrated immediately:

$$\frac{v^2}{2} + \frac{a^2}{\gamma-1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma-1} \tag{199}$$

where the integration constant is chosen by evaluating the left side at $r = \infty$. This is simply the Bernoulli constant.

The Bernoulli equation can be used to relate v and r. Expressing a in terms of v, we find

$$\frac{v^2}{2} + \frac{a_\infty^2}{\gamma-1}\left(\frac{\dot{m}}{4\pi r^2 |v| \rho_\infty}\right)^{\gamma-1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma-1} \tag{200}$$

It is very instructive to look at the case $\gamma = 1.5$, which has a simple, explicit solution:

$$r = \frac{2\left[a_\infty^2(\dot{m}/\pi\rho_\infty|v|)^{1/2} - GM\right]}{4a_\infty^2 - v^2} \tag{201}$$

This expression for r has a singular denominator, and as a consequence has two types of solution. The first has $|v| < 2a_\infty$ everywhere. The velocity goes to zero at large distances, increases as r decreases, and reaches a constant value $|v_0|$ satisfying

$$\dot{m} = \pi\rho_\infty (GM)^2 \frac{|v_0|}{a_\infty^4} \tag{202}$$

as $r \to 0$. Since $|v_0|$ approaches a constant, the density increases sharply as $1/r^2$ at small values of r. These solutions, which are characterized by a limited range of velocity and a large density are known as *settling solutions*.

There are many settling solutions,since $\dot{m}$ is not uniquely determined. If we now regard v_0 as a free parameter, as $|v_0|$ increases from below, approaching $2a_\infty$, the associated mass accretion rate $\dot{m}$ also increases. In the limiting case $|v_0| = 2a_\infty$, $\dot{m}$ reaches its maximum value,

$$\dot{m}_{max} = 2\pi\rho_\infty (GM)^2 a_\infty^{-3} \tag{203}$$

and the singular behavior simply vanishes from equation (201)! We find

$$r = \frac{2GM}{(|v| + 2a_\infty)(|v| + \sqrt{2|v|a_\infty})} \tag{204}$$

This is well-behaved for $|v| = 2a_\infty$. In fact, there is no reason to stop there. The solution extends to infinite $|v|$ as $r \to 0$ and the flow approaches a pure free-fall,

$$v^2 \simeq \frac{2GM}{r} \tag{205}$$

What is the significance of $|v| = 2a_\infty$? When $|v| = 2a_\infty$, equations (204) an (199) imply $a = 2a_\infty$ at the same point. In other words, the point $|v| = 2a_\infty$ is the sonic point $M^2 = 1$. We have shown that for $\dot{m} < \dot{m}_{max}$, the flow remains subsonic everywhere, but becomes transonic at the maximum accretion rate possible. The flow remains transonic from inside the $M^2 = 1$ sonic radius down to $r = 0$, and can be brought to subsonic levels only through the mediation of a shock wave (for example, the surface of the star).

The $\gamma = 1.5$ example is special only in the sense that it is relatively easy to solve. Settling and transonic solutions are found for all values of γ between 1 and 5/3. In fact, while it is not possible to solve the Bondi accretion problem in any simple way for general γ, we can always obtain the value of $\dot{m}_{max}$. For a polytropic equation of state,

$$dP = \frac{\gamma P}{\rho} d\rho = a^2 d\rho \tag{206}$$

Hence,

$$v\frac{dv}{dr} = -a^2 \frac{d\ln\rho}{dr} - \frac{GM}{r^2} = a^2 \frac{d\ln(r^2 v)}{dr} - \frac{GM}{r^2} \tag{207}$$

where mass conservation has been used in the second equality. This leads to the equation

$$v\frac{dv}{dr} = \frac{(v^2/r)(2a^2 - GM/r)}{(v^2 - a^2)} \tag{208}$$

The sonic point S with $v^2 = a^2$ can be crossed only if

$$a_S^2 = \frac{GM}{2r_S} \tag{209}$$

which will ensure that the numerator and denominator vanish together. Combining this with the Bernoulli relation (199) yields

$$a_S^2 = \frac{2a_\infty^2}{5 - 3\gamma} \tag{210}$$

The relation $P = K\rho^\gamma$ implies that

$$\rho_S = \rho_\infty \left(\frac{a_S}{a_\infty}\right)^{2/(\gamma-1)} \tag{211}$$

so that the accretion rate

$$\dot{m} = 4\pi\rho_S r_S^2 a_S = \pi\rho_S (GM)^2 a_S^{-3} \tag{212}$$

can be expressed entirely in terms of a_∞, ρ_∞, and GM, the given parameters of the problem. Upon substitution and simplification,

$$\dot{m} = \alpha^{\alpha/(1-\gamma)} \pi\rho_\infty (GM)^2 a_\infty^{-3}, \qquad \alpha = (5 - 3\gamma)/2. \tag{213}$$

This is the maximum spherical accretion rate possible for any value of γ. The coefficient in front of π varies from $\exp(1.5) = 4.48$ for $\gamma = 1$ to 1 for $\gamma = 5/3$.

Exercise. The center of our galaxy has a black hole of mass 2.6×10^6 solar masses. There is an ambient gas with a number density of about 100 cm^{-3} and a temperature of 10^7. Estimate the Bondi accretion rate onto the Galactic Center black hole by assuming $\gamma = 1.5$. Typically, a black hole might convert 5% of the incoming rest mass energy into radiation. Using this, calculate the expected luminosity of the black hole. The actual luminosity is $\sim 2 \times 10^{33}$ ergs s^{-1}, which should be much less than your result! The Galactic Center is accreting well below its Bondi value.

6.2 The Parker Wind Problem

There is an interesting counterpart to Bondi accretion that involves an outflow from the surface of a star (or perhaps something larger like a cluster or galaxy). This idea was first developed by E. Parker in 1956 and predicted something truly remarkable: the outer layers of the Sun extend throughout and beyond the solar system! The idea is that solar corona can become sufficiently hot near the Sun's surface that the gas can escape to infinity as a cold but rapidly moving (hypersonic) fluid.

The mathematics is very similar to the Bondi problem, but with different boundary conditions. We consider a spherical flow around a central mass M with a sound speed of a_0 at the solar surface r_0. Let the flow velocity at infinity be v_∞ (assuming the gas can in fact escape!). Our Benoulli equation may either be written

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_0^2}{\gamma - 1} - \frac{GM}{r_0} \tag{214}$$

or

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{v_\infty^2}{2} \tag{215}$$

which means that

$$v_\infty^2 = 2\left(\frac{a_0^2}{\gamma - 1} - \frac{GM}{r_0}\right) \tag{216}$$

A nearly isothermal gas can have a temperature corresponding to just a small fraction of the formal escape velocity and still become unbound.

At the sonic point of the outflow, $a_S^2 = GM/2r_S$ still holds as before, and Bernoulli's equation gives

$$a_S^2 = \frac{\gamma - 1}{5 - 3\gamma} v_\infty^2 \tag{217}$$

But since the velocity must rise monotonically, $a_S < v_\infty$, and this means that $\gamma < 3/2$ for a wind. For accretion, recall that $\gamma \leq 5/3$.

The outflow rate from the surface of the star can be determined in just the same way that we determined the mass accretion rate for the Bondi problem: go to the sonic point and evaluate

$$\dot{m} = 4\pi r_S^2 \rho_S a_S \tag{218}$$

with

$$\rho_S = \left(\frac{a_S}{a_0}\right)^{\frac{2}{\gamma - 1}} \tag{219}$$

This gives

$$\dot{m} = \pi \alpha^{\alpha/2} G^2 M^2 \rho_0 a_0^{-2/\gamma - 1} v_\infty^{5 - 3\gamma/\gamma - 1} \tag{220}$$

with $\alpha = (\gamma - 1)/(5 - 3\gamma)$.

The effects of a magnetic field in a stellar wind have been studied by Weber and Davis (1967 ApJ, 148, 217). The problem is complex because there is not one, but *three* critical points! This is because there are three types of propagating disturbances: fast waves, slow waves, and Alfvén waves. Each has its own associated critical point.

We will not present the details here, but refer to the reader to this clearly written and comprehensive paper. The astrophysical significance of this solution is that it shows that magnetic torques can, over the lifetime of the Sun exert a significant angular momentum loss.

7 Accretion in Disks

7.1 Introduction

In any realistic astrophysical system, accreting gas does not simply fall onto a central star in a radial streamline. There is always some relative angular momentum between the gas and star. The gas is diverted and slowed by angular momentum conserving Corilois forces as it approaches the center, and the "pile-up" ultimately takes the form of a dissipative disk. The disk is a reservoir of angular momentum. In order for further accretion to occur, the fluid in the disk must rid itself of angular momentum, and in the process, it will be able to gradually spiral into the center of the disk. The angular momentum is conveyed outward as the material spirals inward. This extended process is really what we mean by an accretion disk. A state of minimum energy is achieved when all the material has collected at the center, and all the angular momentum is at infinity, contained in a vanishingly small fraction of the mass.

In this section, we will see how this happens. Magnetic fields, it will emerge, are crucial.

Exercise. Show that if the surface density of a disk is proportional to $R^{-5/2}$ at large distances from the origin, the angular momentum is logarithmically infinite (i.e. dominated by large R), but that the total mass is within radius R falls rapidly with $R \to \infty$. How rapidly?

7.2 Energy and angular momentum fluxes

To begin our study, we need to have an equation expressing total energy conservation with magnetic fields. Begin with the induction equation written in Cartesian index notation:

$$\partial_t B_i + v_j \partial_j B_i = -B_i \boldsymbol{\nabla \cdot v} + B_j \partial_j v_i \tag{221}$$

Multiply by B_i and sum over i:

$$\partial_t \left(\frac{B^2}{2} \right) + v_j \partial_j \left(\frac{B^2}{2} \right) = -B^2 \boldsymbol{\nabla \cdot v} + B_i B_j \partial_j v_i \tag{222}$$

After a rearrangement of the second term on the left (show!):

$$\partial_t \left(\frac{B^2}{2} \right) + \partial_j \left(\frac{B^2 v_j}{2} \right) = -\frac{B^2}{2} \boldsymbol{\nabla \cdot v} + B_i B_j \partial_j v_i \tag{223}$$

Next, consider the equation of motion:

$$\rho \partial_t v_i + \rho v_j \partial_j v_i = -\partial_i (P + B^2/2\mu_0) - \rho \partial_i \Phi + B_j \partial_j B_i / \mu_0 \tag{224}$$

Multiplying by v_i, summing over i and performing exactly the same kind of manipulations that we did for the induction equation gives us:

$$\partial_t \left(\frac{\rho v^2}{2} \right) + \partial_j \left[\left(\frac{\rho v^2}{2} + P + \Phi \right) v_j + \frac{B^2 v_j}{2\mu_0} - \frac{v_i B_i B_j}{\mu_0} \right] = \mathrm{RHS} \tag{225}$$

where RHS, the right hand side of the equation, is

$$RHS = P \boldsymbol{\nabla \cdot v} + (\partial_i \Phi) \boldsymbol{\nabla \cdot} (\rho \boldsymbol{v}) + \frac{B^2}{2\mu_0} \boldsymbol{\nabla \cdot v} - \frac{1}{\mu_0} B_i B_j \partial_j v_i \tag{226}$$

The last two terms on the right side of the RHS equation are the same as the right side of equation (223) with a minus sign. If we now combine equation (223) with the above equation, after some algebraic simplification (which you should show), we find

$$\partial_t \left(\frac{\rho v^2}{2} + \rho \Phi + \frac{B^2}{2\mu_0} \right) + \boldsymbol{\nabla \cdot} \left[\left(\frac{\rho v^2}{2} + \rho \Phi + P \right) \boldsymbol{v} - \frac{1}{\mu_0} (\boldsymbol{v} \times \boldsymbol{B}) \times \boldsymbol{B} \right] = P \boldsymbol{\nabla \cdot v} \tag{227}$$

This equation is still not quite exact, since we have ignored dissipation and radiation (these would appear as loss terms on the right side of the equation), but what we are really interested in is the mechanical energy flux. *This,* we *have* calculated correctly on the left side of the equation.

The contribution of the magnetic field to the energy flux is the double cross product, the final term in the divergence on the left side. The is just the classical Poynting flux, $\boldsymbol{E} \times \boldsymbol{B}/\mu_0$, the general expression for the flux of electromagnetic energy. The effects of the magnetic field could have been guessed in advance: add the magnetic energy density $B^2/2\mu_0$ to the hydrodynamic energy density, and add the Poynting flux to the hydrodynamic energy flux. Leave the rest untouched.

In classical disk theory, the dominant terms in the energy flux are taken to be

$$\left(\frac{\rho v^2}{2} + \rho\Phi\right)\boldsymbol{v} - \frac{1}{\mu_0}(\boldsymbol{v} \times \boldsymbol{B})\times\boldsymbol{B} \tag{228}$$

i.e., the pressure term is ignored. Nominally, this is done because the disk's rotational energy is larger than its thermal energy. We will come back to this nontrivial point at the end, after we have a better understanding of the issues that are involved.

Let us consider next the angular momentum flux, a simpler task. The exact azimuthal equation of motion may be written

$$\rho\partial_t(Rv_\phi) + \rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}(Rv_\phi) = -\partial_\phi(P + B^2/2\mu_0) + (1/\mu_0)\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}(RB_\phi) \tag{229}$$

This may be written in the form

$$\partial_t(\rho Rv_\phi) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[R\left(\rho v_\phi\boldsymbol{v} - \frac{B_\phi\boldsymbol{B}}{\mu_0} - \left(P + \frac{B^2}{2\mu_0}\right)\boldsymbol{e_\phi}\right)\right] = 0 \tag{230}$$

The non-toroidal component of the angular momentum flux is then conserved. If we integrate over z, and regard the velocities, both kinematic and Alfvénic, as density weighted averages, the statement of angular momentum conservation becomes

$$R^2\Sigma\,(v_\phi v_R - v_{A\phi}v_{AR}) = \text{constant}. \tag{231}$$

where Σ is $\int \rho dz$, the height-integrated column density, and each of the v_A terms represents an Alfvén velocity component. (Do you understand the leading factor of R^2?)

7.3 Fluctuations

Accretion disks are believed to be turbulent, so that each flow variable is considered to have a mean part plus a fluctuating part. The fluctuation has zero for its time-averaged value.

The azimuthal velocity is written

$$v_\phi = R\Omega + \delta v_\phi \tag{232}$$

where $R\Omega$ represents the Keplerian velocity and δv_ϕ the fluctuation with zero mean value. The δ-notation is meaningful: these are true Eulerian fluctuations, taken at fixed spatial location. We consider $R\Omega \gg \delta v_\phi$. The radial velocity is written

$$v_R = \bar{v} + \delta v_R \tag{233}$$

where $\bar{v}$ is the slow inward drift velocity of the gas that results from orbital decay, and δv_R is the (in this case) much larger fluctuation. δv_R and δv_ϕ are taken to be comparable; in practice they are both about 10% of the sound speed. In principle, P, ρ, Σ, and $\boldsymbol{B}$ all have mean and fluctuating components as well, but there is nothing to be gained by an explicit decomposition for these quantities. The Alfvén velocities are taken to be of the same order as the fluctuating velocity components.

In a steady-state model, the mass accretion rate $\dot{M}$ is a constant. This implies

$$\dot{M} = -2\pi R\Sigma\bar{v} = \text{constant.} \tag{234}$$

Now, the conserved angular momentum flux (231) may be written

$$R^2\Sigma\left[(\bar{v} + \delta v_R)(R\Omega + \delta v_\phi) - v_{AR}v_{A\phi}\right] = \text{constant,} \tag{235}$$

or

$$-R^2\Omega\frac{\dot{M}}{2\pi} + R^2\Sigma W_{R\phi} = \text{constant} \tag{236}$$

where we have introduced the quantity

$$W_{R\phi} \equiv \langle \delta v_R\, \delta v_\phi - v_{AR}v_{A\phi}\rangle. \tag{237}$$

$W_{R\phi}$ is proportional to the turbulent stress, and the averaging indicated by the angle brackets should be thought of as a time average.

What about the constant on the right side of equation (236)? A good question! If the inner edge of disk abuts a hard surface, there is some logic to adopting the boundary condition that $W_{R\phi}$ vanish at the inner edge $R = R_0$. That, in any case, is what classical disk theory does. If the inner edge of the disk is near the Schwarzschild radius R_S of a black hole, the arguments are (even!) more vague. In general relativity, one finds that inside of a radius of order R_S the disk is hydrodynamically unstable (this is rigorously true), with the expected consequence that a dramatic change occurs (a guess!). The disk goes from a state of rotational support outside of this "ISCO" (Innermost Stable Circular Orbit), to a state of plunging freefall inside the ISCO. Such an interior flow cannot support a stress at its boundary, and therefore once again the stress must vanish at some R_0.

What to make of this? Numerical simulations tend to show nothing dramatic at the ISCO, but this is currently an ongoing point of controversy.

Much is at stake here: knowledge of the inner edge of a black hole accretion disk is essential if proposed tests of strong field gravity are to be viable. (See Noble, Krolik, & Hawley 2010 Astrophysical Journal, 711, 959 and references therein.) We will assume, however, following the classical approach, that the stress vanishes at some radius R_0.

If $W_{R\phi}$ vanishes at $R = R_0$, then angular momentum conservation gives

$$W_{R\phi} = \frac{\dot{M}\Omega}{2\pi\Sigma}\left[1 - \left(\frac{R_0}{R}\right)^{1/2}\right] \tag{238}$$

where we have assumed a Keplerian law for $\Omega \propto 1/R^{3/2}$. With knowledge of Σ, the R dependence of the turbulent stress is known.

7.4 Energy Loss

The radial component of the energy flux is

$$(\bar{v} + \delta v_R)\left[\rho\frac{(R\Omega + \delta v_\phi)^2}{2} + \rho\Phi\right] + \frac{1}{\mu_0}\boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{239}$$

Next, we height integrate and time average, just as before. If we now expand, retain the largest terms, and use $-\Phi = GM/R = R^2\Omega^2$, the energy flux reduces to

$$\frac{\dot{M}}{4\pi}R^2\Omega^2 + \Sigma R^2\Omega W_{R\phi}. \tag{240}$$

Finally, we substitute for $W_{R\phi}$ from the angular momentum result (238). The energy flux is then

$$F_E \equiv \frac{3R\Omega^2\dot{M}}{4\pi}\left(1 - \frac{2}{3}\frac{R_0^{1/2}}{R^{1/2}}\right) \tag{241}$$

But this is *not* a conserved flux! Its divergence is

$$\frac{1}{R}\frac{d(RF_E)}{dR} = -\frac{3}{4\pi}\frac{GM\dot{M}}{R^3}\left[1 - \left(\frac{R_0}{R}\right)^{1/2}\right] \tag{242}$$

In other words there is an energy loss of

$$\frac{1}{2} \times \frac{1}{R}\frac{d(RF_E)}{dR} = -\frac{3}{8\pi}\frac{GM\dot{M}}{R^3}\left[1 - \left(\frac{R_0}{R}\right)^{1/2}\right] \tag{243}$$

from *each side* of the disk. If this loss of mechanical energy is radiated thermally like a black body, the surface temperature of the disk is

$$T_S^4(R) = \frac{3}{8\pi\sigma}\frac{GM\dot{M}}{R^3}\left[1 - \left(\frac{R_0}{R}\right)^{1/2}\right] \tag{244}$$

where σ is the Stefan-Boltzmann constant 5.67×10^{-8} in MKS units. The total luminosity of the disk is $2\pi R_0 F_E(R_0)$, since there is an outward energy flux $F_E(R_0)$ at the inner edge, and none of the energy makes it to infinity. This gives a total disk luminosity of

$$L = \frac{GM\dot{M}}{2R_0} \tag{245}$$

precisely one-half the binding energy of the innermost orbit.

The characteristic disk spectrum was shown by Donald Lynden-Bell to be given by a power law of the form $\nu^{1/3}$. Here is a crude argument for how this emerges.

We have seen that for $R \gg R_0$, the surface temperature T_S is proportional to $R^{-3/4}$. But the surface temperature at radius R is also directly proportional to the characteristic frequency emitted at radius R, since kT_S is of order $h\nu$. So the characteristic frequency at radius R is proportional to $R^{-3/4}$. Another was to say this is that given a particular frequency ν, the *characteristic radius* at which this frequency is typically emitted is proportional to $\nu^{-4/3}$. Therefore, the area of the disk associated with this frequency goes like $\nu^{-8/3}$. A local blackbody spectrum with a frequency $h\nu \simeq kT$ is proportional to ν^3, giving a net result $\nu^3 \times R^2$ of $\nu^{1/3}$.

Unhappy with that argument? OK, back to mathematics. Prove that the spectrum is "bel et bien"

$$F_\nu = \frac{8\pi h\nu^3}{c^3}\int_{R_0}^{\infty} \frac{2\pi R\, dR}{\exp(h\nu/kT_S(R)) - 1} \tag{246}$$

and derive the $\nu^{1/3}$ result yourself for the case $T_S \propto R^{-3/4}$.

Equation (244) is the fundamental equation of classical accretion disk theory. For its existence, it is essential that the quantity $W_{R\phi}$ exists, and for that we need to understand why there are correlations between the radial and azimuthal fluctuations in the velocity and magnetic field. This is the subject of the next section.

8 The Magnetorotational Instability

The magnetorotational instability is one of the most important and remarkable processes in astrophysical MHD. It allows an (almost!) arbitrarily weak magnetic field to completely disrupt a differentially rotating system like a Keplerian disk. Let us see how it works in detail.

8.1 Local Disk Behavior

In what is known as the local limit, we consider the governing equations in the limit that R approaches infinity, but Ω remains finite. This means that such quantities as v_R^2/R or B_ϕ^2/R may be neglected, but v_ϕ^2/R cannot. We allow for the velocities and magnetic fields to have derivatives on scales much smaller than R, so their derivative cannot be neglected. In essence we drop all curvature terms, *except those associated with rotation.*

For example the radial equation of motion is

$$\rho \frac{Dv_R}{Dt} - \rho \frac{v_\phi^2}{R} = -\frac{\partial}{\partial R}\left(P + \frac{B^2}{2\mu_0}\right) - \rho \frac{\partial \Phi}{\partial R} + \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla} B_R \tag{247}$$

Notice that the term in B_ϕ^2/R has been dropped! We have also use the Lagrangian derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}) \tag{248}$$

The azimuthal equation of motion is

$$\rho \frac{Dv_\phi}{Dt} + \frac{\rho v_R v_\phi}{R} = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla} B_\phi \tag{249}$$

We will work with velocity u_ϕ which is defined as $v_\phi - R\Omega$, that is the true velocity v_ϕ with the local Keplerian velocity subtracted. For consistency of notation, we will also use u_R and u_z, although there is no particular distinction to be made for these variables.

With $R\Omega^2 = \partial\Phi/\partial R$, the radial equation of motion becomes in the local approximation

$$\rho \frac{Du_R}{Dt} - 2\rho\Omega u_\phi = -\frac{\partial}{\partial R}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla} B_R \tag{250}$$

and the azimuthal equation of motion is

$$\rho \frac{Du_\phi}{Dt} + \rho \frac{\kappa^2}{2\Omega} u_R = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(P + \frac{B^2}{2\mu_0}\right) + \frac{1}{\mu_0}\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla} B_\phi \tag{251}$$

Here, κ is the so-called "epicyclic frequency," which may be defined as

$$\kappa^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R} \tag{252}$$

In the absence of a magnetic field and pressure forces, a perturbed fluid element would oscillate around its equilibrium location in an elliptical "epicycle" with a period of κ.

We also need the induction equation. For motion in which $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}$ vanishes, the local radial equation is

$$\frac{DB_R}{Dt} = \boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla} v_R \tag{253}$$

and the local azimuthal equation is (be careful with rotational terms!)

$$\frac{DB_\phi}{Dt} = R\boldsymbol{B}\boldsymbol{\cdot}\boldsymbol{\nabla}(\Omega + u_\phi/R) \tag{254}$$

8.2 Linear Analysis

Consider now the following problem. In the equilibrium state, there is a Keplerian disk with a uniform vertical magnetic field. Such a field exerts no forces. We now make small perturbations in the form of fluid displacements, with space/time dependence $\exp(ikz - i\omega t)$, in the R and ϕ directions, and ask what happens.

The perturbed linear radial equation is

$$-i\omega\delta u_R - 2\Omega\delta u_\phi = \frac{ik}{\mu_0\rho}B_z\delta B_R \tag{255}$$

and the perturbed linear azimuthal equation is

$$-i\omega\delta u_\phi + \frac{\kappa^2}{2\Omega}\delta u_R = \frac{ik}{\mu_0\rho}B_z\delta B_\phi \tag{256}$$

The perturbed induction equations are

$$-i\omega\delta B_R = ikB_z\delta u_R \tag{257}$$

and

$$-i\omega\delta B_\phi = ikB_z\delta u_\phi + \delta B_R\frac{d\Omega}{dR} \tag{258}$$

These are four equations in four unknowns, and a solution exisits only if the following dispersion relation is satisfied:

$$\omega^4 - \omega^2(\kappa^2 + 2k^2v_A^2) + (kv_A)^2(k^2v_A^2 + d\Omega^2/d\ln R) = 0 \tag{259}$$

This is a standard quadratic equation in ω^2, and the exact analysis of its solutions is straightforward. It is not difficult to show that the discriminant of this quadratic (i.e., "$b^2 - 4ac$") is always positive, so that ω^2 is always real. This means that the critical mode between stability $\omega^2 > 0$ and instability $\omega^2 < 0$ occurs at $\omega^2 = 0$. If $d\Omega^2/dR > 0$, then there are no wavenumbers with ω solutions corresponding to $\omega^2 = 0$. On the other hand, if $d\Omega^2/dR < 0$, as it is for essentially all astrophysical disks, then for wavenumbers satisfying

$$(kv_A)^2 < -\frac{d\Omega^2}{d\ln R} \tag{260}$$

instability exists. This is the *magnetorotational instability, or MRI.* In a Keplerian disk, this corresponds to $k^2v_A^2 < 3\Omega^2$.

From the explicit solution of the growth rates from equation (259), the following results can be derived. (Get a pencil and paper and do so):

- The wavenumber of the maximum growth rate is given by

$$k^2v_A^2 = -\left(\frac{1}{4} + \frac{\kappa^2}{16\Omega^2}\right)\frac{d\Omega^2}{d\ln R} \tag{261}$$

  The right side is $15\Omega^2/16$ for a Keplerian disk.

- The maximum growth rate is given by

$$|\omega_{max}| = -\frac{1}{2}\frac{d\Omega}{d\ln R} \tag{262}$$

  This is 0.75Ω for a Keperian disk, an enormously fast rate. In one orbit, linear amplitudes grow by more than factor of 100. Notice that the maximum growth rate is independent of the strength of the magnetic field! The field just sets the lengthscale at which the maximum growth occurs.

- The eigenvector of displacement makes an angle of 45° with respect to outward radial axis, with an azimuthal component pointed along the direction of the shear, and opposite to the sense of global rotation. This gives a strong positive value for the stress tensor, $W_{R\phi}$. In numerical simulations, the Alfvén velocity components of the stress tensor are larger than the ordinary velocity components by a factor of 3 to 4.

A more general linear calculation shows that these results hold for any configuration of magnetic field, as long as there is some component of the vertical field. The MRI is widely believed to be the underlying source of turbulence in accretion disks, though important questions still remain to be understood. Foremost among these is how the MRI works in the relatively cool disks around forming stars.

For more information on the MRI, see the review articles that can be obtained on my webpage, http://www.lra.ens.fr/ balbus