Formation Interuniversitaire de Physique

M2
Magnetohydrodynamics

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1 Fundamentals

1.1 Opening Comment

The behavior of a gas subject to large-scale gravitational and magnetic forces is enormously rich and full of surprises. One of my goals in giving this course is to try to give you, the student encountering the topic for the first time, a sense of both the generality and the depth of the problems we are struggling with. Truly, there is not an area of modern astrophysics that is not touched in some way by the dynamical behavior of gases. Astrophysical gas dynamics is, in the view of your author, the most fundamental component of astrophysics. It is impossible to understand star formation, stellar structure, planet formation, accretion disks, or anything in the early universe without a detailed knowledge of the dynamics of magnetizedgases. So why don't we start?

1.2 Governing Equations

Although the fundamental objects are the atomic particles that comprise our gas, we shall work in the limit in which the matter is regarded as a *nearly* continuous fluid. The fact that this is not *exactly* a continuous fluid manifests itself in many ways, the most important of which is the equation of state of an ideal gas, which depends upon the notion of rapid atomic collisions separated by "long" intervals of time when the atoms are, in essence, free. But more subtle transport effects are also present, like viscosity and thermal conduction, both of which are a consequence of atomic collisions.

One of the most interesting and salient features of astrophysical gases is that they are almost always magnetized. This allows modes of behavior that are absent in an ordinary nonmagnetized gas (e.g. shear waves). This sometimes has profound consequences, especially in rotating systems. The dynamics of magnetized gases is known as magnetohydrodynamics, or MHD for short. The ohmic resistivity of a magnetized gas is another example of a collisional process involving individual particles; in this case one of the particles must be the current carrying electrons.

I shall assume that the reader is familiar with the basic equations of standard hydrodynamics. If not, they[1] may review a standard textbook (my favorite is *Elementary Fluid Dynamics* by D.J. Acheson), or the set of extensive notes I have prepared for my course Hydrodynamics, Instability and Turbulence. We begin with a very brief review.

[1] In these notes, I will use "they" to mean generically "he or she".

1.2.1 Mass Conservation

The statement of mass conservation is expressed by the equation:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0, \tag{1}$$

Here ρ is the mas density and $\boldsymbol{v}$ is the velocity field. The content of this equation is simply that if there is net a mass flux into or out of a fixed volume, the mass within that volume must change accordingly. If the flow is divergence free, the density of an individual fluid element remains constant, and if all fluid elements start somewhere with the same density, the density is everywhere constant.

1.2.2 Newtonian Dynamics

Our second fundamental equation is a statement of Newton's second law of motion, that applied forces cause acceleration in a fluid. The acceleration refers to an individual element of fluid, hence the time derivative is expressed as a *total* derivative, following the path of the element:

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \right] = \boldsymbol{F} \tag{2}$$

where the right side is the sum of the forces on the fluid element.

A fundamental force that is always present acting on a fluid is the pressure. We shall always be working with an ideal gas in this course, and the pressure is then given by the ideal gas equation of state

$$P = \frac{\rho k T}{m} \tag{3}$$

where T is the temperature in Kelvins, k is the Boltzmann constant 1.38×10^{-23} J K^{-1}, and m is the mass per particle. For a fully ionized gas consisting of protons and electons, m is $0.5m_p$, one half of the proton mass (the electron mass being negligible). The quantity kT/m arises often enough that it will be given its own name:

$$c_S^2 \equiv \frac{kT}{m} \tag{4}$$

where the subscript S refers to "sound" for reasons that will become clear later.

The pressure arises from the kinetic energy of the gas particles themselves, which must never be confused with fluid elements. A fluid element is small enough that it has uniquely defined dynamic and thermodynamic attributes (e.g. density and pressure), but large enough to contain a vast number of atoms. A fluid element has a well-defined entropy for example, an atom does not.

There is a very simple relationship between the pressure P and internal energy density $\mathcal{E}$ of an ideal gas:

$$\mathcal{E} = \frac{P}{\gamma - 1}. \tag{5}$$

Here γ is the adiabatic index of the gas. It is equal to 5/3 for single particles, and 7/5 for diatomic molecules.

A pressure exerts a force only if it is not spatially uniform. For example, the pressure force in the x direction on a slab of thickness dx and area $dy\,dz$ is

$$[P(x - dx/2, y, z, t) - P(x + dx/2, y, z, t)]dy\,dz = -\frac{\partial P}{\partial x} dV \tag{6}$$

There is nothing special about the x direction, so the force per unit volume from a pressure is more generally $-\boldsymbol{\nabla} P\, dV$.

Other forces can be added on as needed. One force of obvious importance in astrophysics is gravity. The Newtonian gravitational acceleration $\boldsymbol{g}$ can always be derived from a potential function

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi \tag{7}$$

If the field is from an external source, then Φ is a given function of $\boldsymbol{r}$ and t, otherwise it must be computed along with the evolution of the fluid itself. We shall discuss the problems of self-gravity later in the course.

Another force that we must consider, which will be front and center in this course, arises from the presence of a magnetic field. As we have already noted, magnetic fields allow a gas to behave in ways not allowed when the field vanishes, and the additional degrees of freedom imparted to a gas mean that magnetic forces can be very important even when the field *appears* to be weak! To calculate the magnetic force per unit volume exerted by a magnetic field, start with the Maxwell equation

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mu_0 \boldsymbol{J} + \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{8}$$

The effects of the displacement current are negligible for nonrelativisitic fluids, since they involve time delays associated with light propagation. Hence, the current density is determined by the magnetic field geometry:

$$\boldsymbol{J} = (1/\mu_0)\boldsymbol{\nabla}\times\boldsymbol{B} \tag{9}$$

The Lorentz force per unit volume is $\boldsymbol{J}\times\boldsymbol{B}$, assuming that the gas is everywhere locally neutral.

In the absence of dissipational processes, the equation of motion for a magnetized gas is therefore

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\rho\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} \tag{10}$$

1.2.3 Energetics

The thermal energy behavior of the gas is described by the internal energy loss equation, which is most conveniently expressed in terms of the entropy per particle. The entropy is defined up to an (unimportant) additive constant, and is given by

$$s = \frac{S}{N} = \frac{k}{\gamma - 1}\ln P\rho^{-\gamma} \tag{11}$$

where N is the number of particles, γ is the adiabatic index (equal to $1+2/f$ where f is the number of degrees of freedom of a particle).

Exercise. Derive the above expression from $dE = -PdV + TdS$, $P = \rho kT/m = (\gamma - 1)\mathcal{E}$, $E = \mathcal{E}V$.

The entropy of a fluid element is conserved unless there is a loss or gain of heat from radiative processes or from dissipation. If n is the number of particles per unit volume, then

$$nT\frac{Ds}{Dt} = \frac{P}{\gamma - 1}\frac{D\ln P\rho^{-\gamma}}{Dt} = \text{volume heating rate} \equiv \dot{Q} \tag{12}$$

If there are no radiative losses or gains and no dissipation, as is often the case when the fluid motions are too rapid for heat to escape, the fluid is said to be *adiabatic* and the right side of the above is zero. Note that the internal thermal energy is *not* conserved in an adiabatic fluid because of compression or expansion. As an exercise, the reader should show that c_S^2 satisfies the equation

$$\rho\frac{D}{Dt}\frac{c_S^2}{\gamma - 1} = -P\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{13}$$

for an adiabatic gas. (Use the entropy and mass conservation equations.) The temperature of a fluid element, like the density, remains fixed only if the motions are incompressible.

1.3 The vector "v dot grad v"

The vector $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is more complicated than it appears. In Cartesian coordinates, matters are simple: the x component is just $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_x$, and similar for y, z. But in cylindrical coordinates, say, the radial component of this vector is NOT $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R$, where v_R is the radial velocity component. Rather, we must take care to write

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\nabla}(v_R\boldsymbol{e_R} + v_\phi\boldsymbol{e_\phi} + v_z\boldsymbol{e_z}) \tag{14}$$

where the $\boldsymbol{e_i}$ are unit vectors in their respective directions. In Cartesian coordinates, these unit vectors would be constant, but in any other coordinate system they change with position. You should be able to show that

$$\frac{\partial \boldsymbol{e_R}}{\partial\phi} = \boldsymbol{e_\phi}, \quad \frac{\partial \boldsymbol{e_\phi}}{\partial\phi} = -\boldsymbol{e_R}, \tag{15}$$

and that there are no other unit vector derivatives in cylindrical coordinates. (Do it now. Hint: $\boldsymbol{e_R} = \cos\phi\boldsymbol{e_x} + \sin\phi\boldsymbol{e_y}$, and $\boldsymbol{e_\phi} = -\sin\phi\boldsymbol{e_x} + \cos\phi\boldsymbol{e_y}$.) Thus, the radial component of $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_R - \frac{v_\phi^2}{R}, \tag{16}$$

and the azimuthal component is

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_\phi + \frac{v_R v_\phi}{R} \tag{17}$$

The extra terms are related to centripetal and Coriolis forces, though more work is needed to extract the latter...a piece of it still remains in the gradient term!

1.4 Rotating Frames

It is often useful to work in a frame rotating at a constant angular velocity Ω, perhaps the frame in which an orbiting planet appears at rest around its star. The same rule that applies to ordinary point mechanics applies here as well: add

$$-2\boldsymbol{\Omega}\times\boldsymbol{v} + R\Omega^2\boldsymbol{e_R} \tag{18}$$

to the applied forces operating on a fluid element (the right side of the MHD equation of motion). The first term is the Coriolis force, the second is the centrifugal force, $\boldsymbol{\Omega}$ is in the vertical direction, and all velocities are measured relative to the rotating frame of reference.

1.5 Manipulating the Fluid Equations

For a particular astrophysical problem, working in cylindrical or spherical cooridinates is often the most convenient, but for proving general theorems or identities, Cartesian coordinates are usually the simplest to use. In this case, there is a formalism that makes working with the MHD fluid equations much easier.

The index i, j, or k will represent Cartesian component x, y, or z. Hence v_i means the ith component of v, which may any of the three depending upon what value i is chosen. So v_i is a way to write $\boldsymbol{v}$. The gradient operator $\boldsymbol{\nabla}$ is written ∂_i, in a way that should be self-explanatory.

Next, if an index appears twice, it is understood that it is to be summed over all the values x, y, and z. Hence

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i = A_x B_x + A_y B_y + A_z B_z, \tag{19}$$

and

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v} = (v_i \partial_i) v_j \tag{20}$$

In this last example i is a "dummy index": the actual vector component is represented by j. The dynamical equation of motion in this notation is

$$\rho[\partial_t + (v_i \partial_i)] v_j = -\partial_j P - \rho \partial_j \Phi \tag{21}$$

Sometimes the "rot" (or "curl") operator is needed. For this, we intoduce the Levi-Civita symbol ϵ^{ijk}. It is defined as follows:

- If any of the i, j, or k are equal to one another, then $\epsilon^{ijk} = 0$.
- If $ijk = 123$, 231, or 312, the so-called even permutations of 123, then $\epsilon^{ijk} = +1$.
- If $ijk = 132$, 213, or 321, the so-called odd permutations of 123, then $\epsilon^{ijk} = -1$.

By explicitly writing out each side of the equation, it is straightforward to show that

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \epsilon^{ijk} \partial_i A_j. \tag{22}$$

Here, the vector component is represented by the index k. Don't forget to sum over i and j! ϵ^{ijk} is of course used in the ordinary cross product as well:

$$\boldsymbol{A} \times \boldsymbol{B} = \epsilon^{ijk} A_i B_j. \tag{23}$$

Notice that

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \epsilon^{ijk}A_kB_iC_j \tag{24}$$

which proves that any even permutation of the vectors on the left side of the equation must give the same value, and an odd rearrangement gives the same value with the opposite sign.

A double cross product looks complicated:

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \epsilon^{lkm}A_l(\epsilon^{ijk}B_iC_j) = \epsilon^{mlk}\epsilon^{ijk}A_lB_iC_j. \tag{25}$$

The last equality follows because mlk is an even permutation of lkm. This looks unpleasant, but fortunately there is an identity that saves the day:

$$\epsilon^{mlk}\epsilon^{ijk} = \delta_{mi}\delta_{lj} - \delta_{mj}\delta_{li} \tag{26}$$

where δ_{ij} is the Kronecker delta function (equal to zero if i and j are different, unity if they are the same). One may always prove this by brute force, but an outline of a shorter proof would be to note that the left side has at most one nonvanishing term in its sum, under all circumstances. Moreover, for this one term not to vanish, the index pair (i, j) must be the same *distinct* pair of numbers as (m, l) or (l, m). You can now check that in all cases, the sign $+1$ or -1 always comes out correctly on both sides of the equation. With this identity, our double cross product becomes

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = B_mA_jC_j - C_mA_iB_i = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}). \tag{27}$$

Our final example is to derive an expression for

$$\boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) = \epsilon^{ijk}A_i(\epsilon^{lmj}\partial_lB_m) = \epsilon^{kij}\epsilon^{lmj}(A_i\partial_lB_m) \tag{28}$$

Using our identity (26), this becomes

$$(\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})(A_i\partial_lB_m) = A_i\partial_kB_i - A_i\partial_iB_k = A_i\partial_kB_i - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{29}$$

One consequence of this is a representation of $A_i\partial_kB_i$ in any coordinate system:

$$A_i\partial_kB_i = \boldsymbol{A}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) + (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{30}$$

Another *particularly* important application of (30) is to the Lorentz force expression, something very important for this course. Substituing $\boldsymbol{B}$ for $\boldsymbol{A}$ in the above gives us:

$$(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B} = -\frac{1}{2}\boldsymbol{\nabla}B^2 + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{31}$$

The first term on the right side has the form of a magnetic pressure gradient; the second behaves like a tension force. It depends on the derivative of $\boldsymbol{B}$ along its length, and if the magnitude of $\boldsymbol{B}$ remains fixed, the force must be perpendicular to $\boldsymbol{B}$ itself. The effect of this tension force is profound, allowing a magnetized gas to support shear waves (known as Alfvén waves) that do not exist in a standard, nonmagnetized fluid. In this sense, a magnetized gas behaves more like a solid!

1.6 The Conservation of Vorticity

Let us return, just for the moment, to an unmagnetized fluid. We start with the following identity, which follows immediately from the results of the previous section:

$$\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = \frac{1}{2}\boldsymbol{\nabla} v^2 - (\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} \tag{32}$$

Using this result to replace $(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v}$ in the dynamical equation of motion results in

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla} v^2 - \boldsymbol{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\boldsymbol{\nabla} P - \boldsymbol{\nabla}\Phi \tag{33}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ is known as the vorticity. If we take the curl of this equation and remember that the curl of the gradient vanishes, we find

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho\times\boldsymbol{\nabla}P) \tag{34}$$

Let us once again consider the case where either ρ is constant, or when P is a function only of ρ. In that case, the right hand side vanishes and:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0. \tag{35}$$

To understand what this means, consider a closed linear curve, like a ring, moving with the fluid. The integral $\int \boldsymbol{v}\cdot\boldsymbol{dl}$ around the ring is also $\int \boldsymbol{\omega}\cdot\boldsymbol{dA}$, taken over an area that is bounded by the ring. This is the vorticity flux. How does the vorticity flux change with time as the fluid evolves?

Let us consider more generally a generic equation of the form

$$\frac{\partial \boldsymbol{A}}{\partial t} = \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) + \boldsymbol{\nabla}\Phi \tag{36}$$

where Φ is a potential function. The curl of this equation leads directly to equation (35) for the special case $\boldsymbol{A} = \boldsymbol{v}$, but it is better to retain generality here, because this same equation will be useful when we investigate the behavior of magnetic fields. Expanding the double cross product on the right (which you should be able to do by now!) and regrouping leads to

$$\frac{DA_i}{Dt} = v_j\partial_i A_j + \partial_i\Phi \tag{37}$$

where D/Dt is the standard Lagrangian derivative, and we have, of course, switched over to index notation. We now consider the change in the line integral of the vector field $\boldsymbol{A}$ over a closed curve moving with the fluid itself:

$$\frac{D}{Dt}\oint \boldsymbol{A}\cdot\boldsymbol{dl} = \oint\left[\frac{D\boldsymbol{A}}{Dt}\boldsymbol{\cdot}\boldsymbol{dl} + \boldsymbol{A}\boldsymbol{\cdot}\frac{D\boldsymbol{dl}}{Dt}\right] \tag{38}$$

Hmmm. How interesting. We are taking the derivative of a differential $\boldsymbol{dl}$. Have you ever done that before? Don't panic. The Lagrangian derivative of a line element $\boldsymbol{dl}$ moving with the fluid is just the difference between the fluid's velocity field that is pulling at the two endpoints of the segment $\boldsymbol{dl}$. If $\boldsymbol{dl}$ is the line element at time $t = 0$, and $\boldsymbol{dl'}$ is the same line element an instant later at time $t = \Delta t$, then

$$\boldsymbol{dl'} = \boldsymbol{dl} + \Delta t\ (\boldsymbol{dl}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v}, \tag{39}$$

or

$$\frac{D\boldsymbol{dl}}{Dt} = \frac{\boldsymbol{dl'} - \boldsymbol{dl}}{\Delta t} = (\boldsymbol{dl}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} \tag{40}$$

as $\Delta t \to 0$. This may also be written

$$\frac{Ddl_j}{Dt} = dl_i\partial_i v_j \tag{41}$$

We then have from equation (37)

$$\frac{D\boldsymbol{A}}{Dt}\boldsymbol{\cdot}\boldsymbol{dl} = dl_i\ v_j\partial_i A_j + dl_i\partial_i\Phi \tag{42}$$

and

$$\boldsymbol{A}\boldsymbol{\cdot}\frac{D\boldsymbol{dl}}{Dt} = A_j dl_i\partial_i v_j \tag{43}$$

Adding these last two equations gives

$$\oint\frac{D}{Dt}(\boldsymbol{A}\cdot\boldsymbol{dl}) = \oint dl_i\partial_i(\Phi + v_jA_j) = \oint \boldsymbol{dl}\cdot\boldsymbol{\nabla}(\Phi + v_jA_j) \tag{44}$$

This is a perfect gradient function integrated around a closed curve. Since the beginning and end points are the same, it must vanish. The line integral $\oint \boldsymbol{A}\cdot\boldsymbol{dl}$ is conserved with the fluid. In particular, when $\boldsymbol{A} = \boldsymbol{v}$, the velocity circulation integral along with the vorticity flux surface integral are conserved in the Lagrangian sense, moving with the fluid. We shall see very soon that the same is true for the magnetic field and magnetic flux.

The fact that the integral $\oint \boldsymbol{v}\cdot\boldsymbol{dl}$ around any closed curve in the fluid remains constant as the it flows with the fluid is known as vorticity conservation. Another way to say the same thing is that the field lines of vorticity

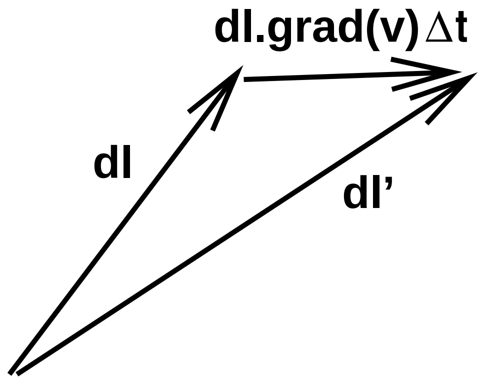

Figure 1: The change of a line element $\boldsymbol{dl}$ in time Δt.

$\boldsymbol{\omega}$ are "frozen" into the fluid. Once again, this is not a completely general fluid result, even if there is no magnetic field. We had to assume either that ρ is constant, or that P and ρ are functionally related, $P = P(\rho)$. (This is called a *barotropic* fluid.)

With the help of our $\epsilon^{ijk}\epsilon^{lmk}$ identity and just a little work, it is quite straightforward to show that the equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{\omega}) = 0 \tag{45}$$

is the same equation as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = +(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\omega}\boldsymbol{\nabla}\cdot\boldsymbol{v} \tag{46}$$

Now, mass conservation implies

$$\frac{D\ln\rho}{Dt} = -\boldsymbol{\nabla}\cdot\boldsymbol{v}, \tag{47}$$

so that our equation becomes

$$\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\frac{D\ln\rho}{Dt} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v}, \tag{48}$$

or

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{1}{\rho}(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\boldsymbol{v} \tag{49}$$

This is a very interesting result! Notice that $\boldsymbol{\omega}/\rho$ satisfies exactly the same equation (41) as the line element $\boldsymbol{dl}$. But, by definition, $\boldsymbol{dl}$ was a small line segment moving with the fluid. So this is a direct way of showing that $\boldsymbol{\omega}/\rho$ moves with the fluid.

Next, consider a flow that is strictly two-dimensional. Then ω has only a z component, and the right side of equation (49) must vanish. Moreover, in equation (47), we replace the density ρ by the surface (or "column") density Σ. We then find instead of (49),

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\Sigma}\right) = 0 \tag{50}$$

This is known as the conservation of potential vorticity. It is an extremely useful and powerful constraint in the study of two-dimensional turbulence, and in studying long wavelength disturbances in planetary atmospheres.

Exercise. Consider purely rotational flow, with the velocity $\boldsymbol{v}$ having only a ϕ component v_ϕ. In general, v_ϕ could depend upon R and z, but show that if vorticity conservation holds, then under steady conditions v_ϕ cannot depend upon z. This is known as *von Zeippel's* theorem.

Exercise. Two-dimensional turbulence in a fluid is never spontaneous, it must always be driven externally. This is *not* true of three dimensional turbulence. Explain this far reaching result in terms of potential vorticity conservation. (Hint: What would happen if we had even a tiny amount of dissipation in a two-dimensional fluid?)

Exerccise. In planetary atmospheres, local disturbances that lose vorticity find their way up to the north pole (or down to the south pole), and settle down as "polar vortex rings." Explain. (Hint: The total vorticity of a disturbance, including the contribution from the planet's rotation plus the intrinsic vorticity within the gas, must be conserved.)

2 Magnetohydrodynamics (MHD)

2.1 Magnetic Forces

We return to magnetic fields. Astrophysical gases are almost always at least partially ionized. This is not too surprising: a glass of distilled water is ionized at the level of one part in 10^7, and salty sea water is much more ionized: it is a very good conductor. A medium can be almost entirely neutral and still behave like a good conductor. All but the coolest and densest astrophysical gases (e.g., protostellar disks) are electrodynamically active.

The Lorentz force per unit volume in the gas is

$$\boldsymbol{F} = \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} \tag{51}$$

where ρ_e is the charge density, $\boldsymbol{E}$ is the electric field, $\boldsymbol{J}$ is the current density, and $\boldsymbol{B}$ is the magnetic field. The gases of interest here are all electrically neutral, so that $\rho_e = 0$. This means that the only part of the Lorentz force that affects the gas is the magnetic part.

We have already encountered the Lorentz force in our discussion of the equation of motion for a magnetized gas:

$$\boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \tag{52}$$

In the last section, we showed that

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = -\frac{1}{2}\boldsymbol{\nabla} B^2 + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} \tag{53}$$

Thus, the dynamical equation of motion for a magnetized gas is

$$\rho \frac{D\boldsymbol{v}}{Dt} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) - \rho \boldsymbol{\nabla}\Phi + \left(\frac{\boldsymbol{B}}{\mu_0}\cdot\boldsymbol{\nabla}\right)\boldsymbol{B} \tag{54}$$

The first magnetic term on the right clearly behaves like a sort of pressure. Magnetic fields lines of force do not like to be squeezed any more than gas molecules do.

The $(\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B}$ term is less obvious. We have noted that it corresponds to a sort of magnetic tension. Notice that it vanishes when the magnetic field does not change along the direction in which the field line itself is oriented. On the other hand, when there are such changes, and the field line is "stretched", the resulting force acts in the direction of restoring the field line back to an unstretched position. In fact, this can be made quantitative: there is a magnetic analogue to waves propagating along an ordinary string that is under tension. In the case of "magnetic strings," these waves are called Alfvén waves.

2.2 Induction Equation

Having introduced the magnetic field, we need to know how it evolves when there are changes in the fluid. The magnetic field adds one more variable to our problem (well, three actually, since there are three components of $\boldsymbol{B}$), so that we need some more equations. The motion of the gas causes charged particles to move relative to one another, and the resulting electrical currents in turn generate new magnetic fields. These affect the currents, that change the fields again, that ... Help. It seems like a complicated mess!

Fortunately there is indeed help in the form of a great simplifying principle: in a pefect conductor, the electric field vanishes. Actually, what we need to say is that in the rest frame of the conductor, the electric field vanishes. In a frame in which the conductor (in our case the conducting gas fluid element) moves, the total Lorentz force, not the electric field, must vanish. In other words,

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0. \tag{55}$$

So even though we have assumed conditions for charge neutrality, there must be an electric field! Hmmm. Wait. If the divergence of this electric field does not vanish, then according to Maxwell (or Coulomb!) there must be a local charge density, and then charge neutrality cannot hold. This certainly looks like looks like a contradiction. Well, guess what? The divergence of the electric field does *not,* in general, vanish. In a moment, we'll come back and explain why this is not really a contradiction, but for the time being let us continue as though we have nothing to worry about.

Faraday's law of induction is

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E} \tag{56}$$

and with $\boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}$, this becomes

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) \tag{57}$$

This is the equation we need to determine the magnetic field. By knowing how the spatial gradients of $\boldsymbol{B}$ are behaving, we may compute how the field evolves in time, thanks to the powerful constraint that the Lorentz force on the charge carriers must vanish.

Notice something quite remarkable: the magnetic field satisfies the same equation as the vorticity. In particular, equation (57) can be recast in the form of equation (36), by "uncurling" it! That means everything we learned about vorticity, in particular that it is frozen in to the fluid, also holds for the magnetic field. *Magnetic flux,* $\int \boldsymbol{B} \cdot \boldsymbol{dA}$*, is conserved as the area moves with the fluid.* But unlike the case of vorticity conservation, which depended upon a restrictive relationship between P and ρ, magnetic flux conservation depends only upon there being no dissipation (i.e., electrical resistance) in the gas. This is generally an excellent approximation.

2.3 Self-consistency

Why don't we have a contradiction with the fact that $\boldsymbol{\nabla} \cdot \boldsymbol{E}$ is not zero? The answer is that while not zero, it is in fact, you know, small. Small?? Don't give me that. That answer is not good enough. How small? Very small indeed: of order v^2/c^2 (c is the speed of light). This, as we will see, is precisely of the same order as the neglected displacement current.

To estimate $\boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{B})$, assume that any magnetic field gradients are as large as they can be (of order $\mu_0 J$), and that J is also as large as it can be, of order the ion charge density times v, $\rho_i v$ (the current density could be much smaller, since it is proportional to the *difference* between ion and electron velocities). Then

$$\boldsymbol{\nabla}\cdot(\boldsymbol{v}\times\boldsymbol{B}) \sim v\mu_0 J \sim \frac{\rho_i}{\epsilon_0}\frac{v^2}{c^2}. \tag{58}$$

That answer, that the divergence of the electric field is of order v^2/c^2 times the ion charge density, really *is* good enough. Not only is it permitted to ignore the divergence of the electric field, it is required! We have already not included the displacement current, and this too is a correction of order v^2/c^2. In this case, if L is a characteristic length and $\partial/\partial t \sim v/L$, then

$$\epsilon_0\mu_0\frac{\partial \boldsymbol{E}}{\partial t} \sim \epsilon_0\mu_0\frac{vE}{L} \sim \epsilon_0\mu_0\frac{v^2 B}{L} \sim \epsilon_0\mu_0^2 v^2 J \tag{59}$$

which is indeed of order $(v^2/c^2)\mu_0 J$. Corrections of order v^2/c^2 are truly relativistic, and we must ignore them to be self-consistently *nonrelativistic.*

A Summary of the Dissipationless Equations of Motion

From now on, we shall drop the subscript "0" on μ_0, and write μ.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \tag{60}$$

$$\rho\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\boldsymbol{v} = -\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu}\right) - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} \tag{61}$$

$$\frac{P}{\gamma - 1}\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\ln P\rho^{-\gamma} = 0 \tag{62}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B}) \tag{63}$$

3 Fundamentals

In this section, a detailed derivation of the fundamental MHD equations is presented. The discussion will be more technical here than in most of the rest of the course, but it is very important to see how the basic governing equations of the subject arise, and much of this material is not so easy to find outside of specialized treatments. I hope the reader will have the patience to read carefully through this section.

In astrophysics, we are very often interested in the MHD behavior of a gas that is almost entirely neutral. This may seem like contradictory, since a neutral gas has no charge carriers, but the key word is "almost." Even a very small population of charge carriers will make the gas magnetized, as we will shortly see.

Consider a gas consisting of neutral particles (predominantly H_2 molecules), electrons, and ions. Each species (denoted by subscript s) is separately conserved, and obeys the mass conservation equation

$$\frac{\partial \rho_s}{\partial t} + \boldsymbol{\nabla \cdot}(\rho_s \boldsymbol{v}_s) = 0 \tag{64}$$

where ρ_s is the mass density for species s and $\boldsymbol{v_s}$ is the velocity. The symbols of the flow quantities (e.g. $\boldsymbol{v}$, ρ, etc.) for the dominant neutral species will henceforth be presented without subscripts.

So far, everything is simple. The dynamical equations become more complicated, since we need to include interactions between the different species. The dynamical equation for the neutral particles is

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho(\boldsymbol{v \cdot \nabla})\boldsymbol{v} = -\boldsymbol{\nabla} P - \rho \boldsymbol{\nabla} \Phi - \boldsymbol{p_{nI}} - \boldsymbol{p_{ne}} \tag{65}$$

where P is the pressure of the neutrals, Φ the gravitational potential and $\boldsymbol{p_{nI}}$ ($\boldsymbol{p_{ne}}$) is the momentum exchange rate between the neutrals and the ions (electrons).

The Details....

Let us examine these last two important terms a little more closely. (Please don't worry about memorizing every last detail. My purpose here is to give you a feeling for all that goes into a calculation like this, and to be able to understand the nomenclature that you will encounter in the literature. You don't have to become an expert in the minutiae of interstellar kinetic theory for this course!) $\boldsymbol{p_{nI}}$ takes the form

$$\boldsymbol{p_{nI}} = n\mu_{nI}(\boldsymbol{v} - \boldsymbol{v_I})\nu_{nI} \tag{66}$$

where n is the number density of neutrals, and μ_{nI} is the reduced mass of an ion–neutral particle pair,

$$\mu_{nI} \equiv \frac{m_I m_n}{m_I + m_n}, \tag{67}$$

m_I and m_n being the ion and neutral mass respectively. ν_{nI} is the collision frequency of a neutral with a population of ions,

$$\nu_{nI} = n_I \langle \sigma_{nI} w_{nI} \rangle. \tag{68}$$

In equation (68), n_I is the number density of ions, σ_{nI} is the cross section for neutral-ion collisions, and w_{nI} is the relative velocity between a neutral particle and an ion. The angle brackets represent an average over all possible relative velocities in the thermal population of particles. Notice that equation (66) has the dimensions of a force per unit volume, and that it is proportional to the velocity difference between the species: if there is no difference in their mean velocities, two population of particles cannot exchange momentum.

Why does the reduce mass μ_{nI} appear? Because the reduced mass *always* appears in any interaction between two individual particles: in the center of mass frame the equations reduce to a single particle equation with the particle mass equal to the reduced mass. In an elastic one-dimensional collision, for example, if v is initial relative velocity of the two interacting particles, then the momentum exchange is $2\mu_{12}v$, where μ_{12} is the reduced mass. (Show this.)

For neutral-ion scattering, we may approximate the cross section σ_{nI} to be geometrical, which means that the quantity in angle brackets will be proportional to $\mu_{nI}^{-1/2}$. The order of the subscripts has no particular significance in either the cross section σ_{nI}, reduced mass μ_{nI}, or relative velocity w_{nI}. But ν_{In} does differ from ν_{nI}: the former is proportional to the neutral density n, the latter to the ion density n_I.

Putting all these definitions together gives

$$\boldsymbol{p_{nI}} = n n_I \mu_{nI} \langle \sigma_{nI} w_{nI} \rangle (\boldsymbol{v} - \boldsymbol{v_I}) \tag{69}$$

In accordance with Newton's third law, this is symmetric with respect to the interchange $n \leftrightarrow I$, except for a change in sign, $\boldsymbol{p_{nI}} = -\boldsymbol{p_{In}}$. All of these considerations hold, of course, for electron-neutral scattering as well. Explicitly, we have

$$\boldsymbol{p_{ne}} = n n_e \mu_{ne} \langle \sigma_{ne} w_{ne} \rangle (\boldsymbol{v} - \boldsymbol{v_e}) \simeq n n_e m_e \langle \sigma_{ne} w_{ne} \rangle (\boldsymbol{v} - \boldsymbol{v_e}). \tag{70}$$

The gas is assumed to be locally neutral, so that $n_e = Z n_i$ where Z is the number of ionizations per ion particle. In a weakly ionized gas, $Z = 1$. The reduced mass μ_{ne} is very nearly equal to the electron mass m_e. The collision rates are given by (see Draine, Roberge, & Dalgarno 1983 ApJ 264, 485 for yet more details) (note, cgs units!):

$$\langle \sigma_{nI} w_{nI} \rangle = 1.9 \times 10^{-9} \text{ cm}^3 \text{ s}^{-1} \tag{71}$$

$$\langle \sigma_{ne} w_{ne} \rangle = 10^{-15} \, (128kT/9\pi m_e)^{1/2} = 8.3 \times 10^{-10} T^{1/2} \text{ cm}^3 \text{ s}^{-1} \tag{72}$$

The electron-neutral collision rate is just the ion geometric cross section times an electron thermal velocity. (The peculiar factor of $(128/9\pi)^{1/2}$ is a detail of the averaging procedure.) But the ion-neutral collision rate is temperature independent, much more beholden to long range induced dipole interactions, and significantly enhanced relative to a geometrical cross section assumption. Even if the ion-neutral rate were determined only by a geometrical cross section, $|\boldsymbol{p_{nI}}|$ would exceed $|\boldsymbol{p_{ne}}|$ by a factor of order $(m_e/\mu_{nI})^{1/2}$. In fact, the dipole enhancement of the ion-neutral cross section makes this factor larger still[2].

In the astrophysical literature, it is common to write the ion-neutral momentum coupling in the form

$$\boldsymbol{p_{In}} = \rho\rho_I \gamma(\boldsymbol{v_I} - \boldsymbol{v}), \tag{73}$$

where γ is the so-called *drag coefficient*,

$$\gamma \equiv \frac{\langle \sigma_{nI} w_{nI} \rangle}{m_I + m_n} \tag{74}$$

and we will use this notation from here on. Numerically, $\gamma = 3 \times 10^{13}$ cm^3 s^{-1} g^{-1} for astrophysical mixtures (Draine, Roberge, & Dalgarno 1983).

We come next to the ions and electrons. The dynamical equations for the ions and electrons are

$$\rho_I \frac{\partial \boldsymbol{v_I}}{\partial t} + \rho_I \boldsymbol{v_I} \cdot \boldsymbol{\nabla} \boldsymbol{v_I} = -\boldsymbol{\nabla} P_I - \rho_I \boldsymbol{\nabla}\Phi + Zen_I \left(\boldsymbol{E} + \boldsymbol{v_I} \times \boldsymbol{B}\right) - \boldsymbol{p_{In}} \tag{75}$$

and

$$\rho_e \frac{\partial \boldsymbol{v_e}}{\partial t} + \rho_e \boldsymbol{v_e} \cdot \boldsymbol{\nabla} \boldsymbol{v_e} = -\boldsymbol{\nabla} P_e - \rho_e \boldsymbol{\nabla}\Phi - en_e \left(\boldsymbol{E} + \boldsymbol{v_e} \times \boldsymbol{B}\right) - \boldsymbol{p_{en}}, \tag{76}$$

respectively. e will *always* denote the *positive* charge of a proton, the absolute value of the electron charge, 1.602×10^{-19} Coulombs or 4.803×10^{-10} esu.[3]

[2] I should be a little bit more careful. The statement that $|\boldsymbol{p_{ne}}|$ is larger than $|\boldsymbol{p_{nI}}|$ by a factor of $(m_e/\mu_{nI})^{1/2}$ assumes that the velocity differences $\boldsymbol{v} - \boldsymbol{v_e}$ and $\boldsymbol{v} - \boldsymbol{v_I}$ do not introduce any mass dependencies, which is generally true.

[3] Beware: esu units are still commonly used in the astrophysical literature! You should become comfortable with them.

For a weakly ionized gas, the Lorentz force and collisional terms dominate in each of the latter two equations. Comparison of the magnetic and inertial forces, for example, shows that the latter are smaller than the former by the ratio of the proton or electron gyroperiod to a macroscopic flow crossing time. Thus, to an excellent degree of approximation,

$$Zen_I\,(\boldsymbol{E}+\boldsymbol{v_I}\times\boldsymbol{B})-\boldsymbol{p_{In}}=0, \tag{77}$$

and

$$-en_e\,(\boldsymbol{E}+\boldsymbol{v_e}\times\boldsymbol{B})-\boldsymbol{p_{en}}=0. \tag{78}$$

The sum of these two equations gives

$$\boldsymbol{J}\times\boldsymbol{B}=\boldsymbol{p_{In}}+\boldsymbol{p_{en}} \tag{79}$$

where charge neutrality $n_e = Zn_I$ has been used, and we have introduced the current density

$$\boldsymbol{J}\equiv en_e(\boldsymbol{v_I}-\boldsymbol{v_e}). \tag{80}$$

The equation for the neutrals becomes

$$\rho\frac{\partial\boldsymbol{v}}{\partial t}+\rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}=-\boldsymbol{\nabla}P-\rho\boldsymbol{\nabla}\Phi+\boldsymbol{J}\times\boldsymbol{B} \tag{81}$$

Due to collisional coupling, the neutrals are subject to the magnetic Lorentz force just as though they were a gas of charged particles. It is not the magnetic force *per se* that changes in a neutral gas. As well shall presently see, it is the inductive properties of the gas.

Let us return to the force balance equations for the electrons:

$$-en_e\,(\boldsymbol{E}+\boldsymbol{v_e}\times\boldsymbol{B})-\boldsymbol{p_{en}}=0. \tag{82}$$

After division by $-en_e$, this may be expanded to

$$\boldsymbol{E}+[\boldsymbol{v}+(\boldsymbol{v_e}-\boldsymbol{v_I})+(\boldsymbol{v_I}-\boldsymbol{v})]\times\boldsymbol{B}+\frac{m_e\nu_{en}}{e}\,[(\boldsymbol{v_e}-\boldsymbol{v_I})+(\boldsymbol{v_I}-\boldsymbol{v})]=0, \tag{83}$$

where we have introduced the collision frequency of an electron in a population of neutrals:

$$\nu_{en}=n\langle\sigma_{ne}w_{ne}\rangle. \tag{84}$$

We have written the electron velocity $\boldsymbol{v_e}$ in terms of the dominant neutral velocity $\boldsymbol{v}$ and the key physical velocity differences of our problem. It has already been noted that in equation (79), $\boldsymbol{p_{en}}$ is small compared with $\boldsymbol{p_{In}}$, provided that the velocity difference $|\boldsymbol{v_e}-\boldsymbol{v}|$ is not much larger than $|\boldsymbol{v_I}-\boldsymbol{v}|$. As we argued earlier, the $\boldsymbol{p_{en}}$ term in equation (79) is small relative to $\boldsymbol{p_{In}}$:

$$\boldsymbol{J}\times\boldsymbol{B}\simeq\boldsymbol{p_{In}}=nn_I\mu nI(\boldsymbol{v_I}-\boldsymbol{v})\nu_{nI}. \tag{85}$$

It then follows that the final term in equation (83)

$$\frac{m_e \nu_{en}}{e}(\boldsymbol{v_I} - \boldsymbol{v}),$$

which is proportional to $\boldsymbol{J} \times \boldsymbol{B}$, becomes small compared with the third term

$$(\boldsymbol{v_e} - \boldsymbol{v_I}) \times \boldsymbol{B},$$

which also proportional to $\boldsymbol{J} \times \boldsymbol{B}$, by a factor of order $(m_e/\mu_{In})^{1/2}$. These simplifications allow us to write the electron force balance equation as

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{J} \times \boldsymbol{B}}{e n_e} - \frac{\boldsymbol{J}}{\sigma_{cond}} + \frac{(\boldsymbol{J} \times \boldsymbol{B}) \times \boldsymbol{B}}{\gamma \rho \rho_I} = 0, \tag{86}$$

where the electrical conductivity has been defined as

$$\sigma_{cond} \equiv \frac{e^2 n_e}{m_e \nu_{en}} \tag{87}$$

The associated resistivity η is

$$\eta = \frac{1}{\mu_0 \sigma_{cond}}, \tag{88}$$

which has units of $\mathrm{m^2\ s^{-1}}$. Numerically (e.g. Blaes & Balbus 1994 ApJ, 421, 163; Balbus & Terquem 2001, ApJ, 552, 235):

$$\eta = 0.0234 \left(\frac{n}{n_e} \right) T^{1/2}\ \mathrm{m^2\ s^{-1}} \tag{89}$$

Equation (86) is a general form of Ohm's law for a moving, multiple fluid system.

Next, we make use of two of Maxwell's equations. The first is Faraday's induction law:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{90}$$

We substitute $\boldsymbol{E}$ from equation (86) to obtain an equation for the self-induction of the magnetized fluid,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times \left[\boldsymbol{v} \times \boldsymbol{B} - \frac{\boldsymbol{J} \times \boldsymbol{B}}{e n_e} + \frac{(\boldsymbol{J} \times \boldsymbol{B}) \times \boldsymbol{B}}{\gamma \rho \rho_I} - \frac{\boldsymbol{J}}{\sigma_{cond}} \right] \tag{91}$$

It remains to relate the current density $\boldsymbol{J}$ to the magnetic field $\boldsymbol{B}$. This is accomplished by the second Maxwell equation,

$$\mu_0 \boldsymbol{J} = \boldsymbol{\nabla}\times\boldsymbol{B} + \frac{\partial \boldsymbol{E}}{\partial t} \tag{92}$$

The final term in the above is the displacement current, and it may be ignored. Indeed, since we have not, and will not, use the "Gauss's Law" equation

$$\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{E} = (e/\epsilon_0)(Zn_I - n_e), \tag{93}$$

we *must not* include the displacement current. In Appendix B, we show that departures from charge neutrality in $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{E}$ and the displacement current are both small terms that contribute at the same order: v^2/c^2. These must both be self-consistently neglected in nonrelativisitc MHD. (The final Maxwell equation $\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{B} = 0$ adds nothing new. It is automatically satisfied by equation (90), as long as the initial magnetic field satifies this divergence free condition.) These considerations imply

$$\boldsymbol{J} = \frac{1}{\mu_0}\boldsymbol{\nabla}\times\boldsymbol{B} \tag{94}$$

for use in equation (91).

To summarize, the fundamental equations of a weakly ionized fluid are mass conservation of the dominant neutrals (eq.[64])

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho \boldsymbol{v}) = 0, \tag{95}$$

the equation of motion (eq. [81] with [94])

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}P - \rho\boldsymbol{\nabla}\Phi + \frac{1}{\mu_0}(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}, \tag{96}$$

and the induction equation (eq. [91] with [88] and [94])

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times\left[\boldsymbol{v}\times\boldsymbol{B} - \frac{(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}}{\mu_0 e n_e} + \frac{[(\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}]\times\boldsymbol{B}}{\mu_0\gamma\rho\rho_I} - \frac{\boldsymbol{\nabla}\times\boldsymbol{B}}{\mu_o\sigma_{cond}}\right] \tag{97}$$

It is only natural that the reader should be a little taken aback by the sight of equation (97). Be assured that it is rarely, if ever, needed in full generality: almost always one or more terms on the right side of the equation may be discarded. When only the induction term $\boldsymbol{v}\times\boldsymbol{B}$ is important, we refer to this regime as *ideal* MHD. The three remaining terms on the right are the nonideal MHD terms.

To get a better feel for the relative importance of the nonideal MHD terms in equation (91), we denote the terms on the right side of the equation, moving left to right, as I (induction), H (Hall), A (ambipolar diffusion), and O (Ohmic resistivity). We will always be in a regime in which the presence of the induction term is not in question. More interesting is the relative importance of the nonideal terms. The explicit dependence of A/H and O/H in terms of the fluid properties of a cosmic gas has been worked out by Balbus & Terquem (2001):

$$\frac{A}{H} = Z\left(\frac{9\times10^{12}\ \mathrm{cm}^{-3}}{n}\right)^{1/2}\left(\frac{T}{10^3\,\mathrm{K}}\right)^{1/2}\left(\frac{v_A}{c_S}\right) \tag{98}$$

and

$$\frac{O}{H} = \left(\frac{n}{8\times10^{17}\ \mathrm{cm}^{-3}}\right)^{1/2}\left(\frac{c_S}{v_A}\right) \tag{99}$$

Here n is the total number density of all particles, T is the kinetic temperature, v_A is the so-called *Alfvén velocity* (much more about this quantity will come later!),

$$\boldsymbol{v_A} = \frac{\boldsymbol{B}}{\sqrt{\mu_0\rho}} \tag{100}$$

and c_S is the isothermal speed of sound,

$$c_S^2 = 0.429\frac{kT}{m_p} \tag{101}$$

where k is the Boltzmann constant and m_p the mass of the proton. The coefficient 0.429 corresponds to a mean mass per particle of $2.33m_p$, appropriate to a molecular gas with a 10% helium admixture.

As reassurance that the fully general nonideal MHD induction equation is not needed for our purposes, note that equations (98) and (99) imply that for all three nonideal MHD terms to be comparable, $T \sim 10^8$ K! Obviously this is not a weakly ionized regime. In figure (2), we plot the domains of relative dominance of the nonideal MHD terms in the nT plane.

Our emphasis of the relative ordering of the nonideal terms in the induction equation should not obscure the fact that ideal MHD is often an excellent approximation, even when the ionization fraction is $\ll 1$. For example, the ratio of the ideal inductive term to the ohmic loss term is given by the Lundquist number

$$\boldsymbol{\ell} = \frac{v_A H}{\eta} \tag{102}$$

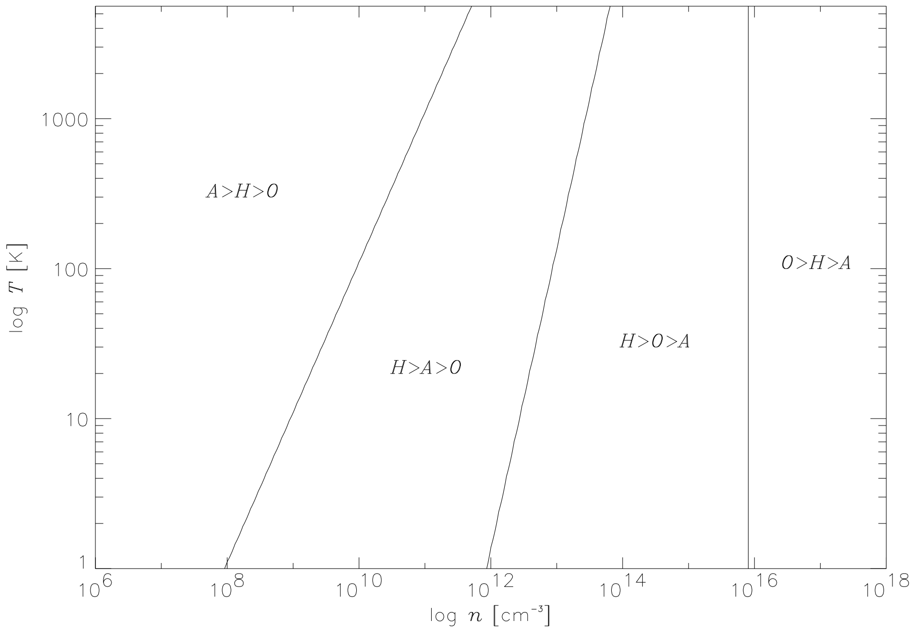

Figure 2: Parameter space for nonideal MHD. The curves correspond to the case $v_A/c_S = 0.1$. (From Kunz & Balbus 2004, MNRAS, 348, 355.)

where H is a characteristic gradient length scale. To orient ourselves, let us consider the case of a protostellar disk and set $H = 0.1R$, where R is the radial location in the disk. (This would correspond to H being about the vertical thickness of the disk.) Then $\boldsymbol{\ell}$ is given by

$$\boldsymbol{\ell} \simeq 2.5(n_e/n)(v_A/c_S)R_{cm},$$

R_{cm} being the radius in centimeters. In other words, the critical ionization fraction at which $\boldsymbol{\ell} = 1$ is about

$$(n_e/n)_{crit} = 0.4(c_S/v_A)R_{cm}^{-1} \sim 10^{-13}(c_S/10v_A)$$

at $R = 1$ AU. The actual ionization fraction at this location may be above or below this during the course of the solar systems evolution, but the point worth noting here is that R_{cm} is a large number for a protostellar disk! Ionization fractions far, far below unity can render an astrophysical gas a near perfect electrical conductor. It therefore makes a great deal of sense to begin by examining the behavior of an ideal MHD fluid.

Exercise. Show that the Lorentz force may be written

$$\boldsymbol{J} \times \boldsymbol{B} = \partial_i \left(\frac{B_i B_j}{\mu} - \delta_{ij} \frac{B^2}{2\mu} \right). \tag{103}$$