Formation Interuniversitaire de Physique

Module :
Hydrodynamics

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TO LEARN: ... Nonlinear Classical Hydro

— From the final blackboard of R. P. Feynman

Contents

References

In this course, we will draw very heavily on the following text:

Acheson, D.J. 1990, *Elementary Fluid Dynamics* (Clarendon Press: Oxford)

This is an excellent book, with a just the right mixture of mathematics, physical discussion, and historical perspective. The writing is exceptionally clear. We will be following the text throughout the course, though I will occasionally deviate both in the order in which material is presented, as well as with the choice of some subject matter. I strongly recommend that students purchase this text. It is available online from amazon.com.

Other recommended references:

Lighthill, J. 1978, *Waves in Fluids* (Cambridge University Press: Cambridge) A truly classic text written by a fluid theorist of great renown. The discussion of the physical meaning of formal wave theory is both deep and interesting. When Lighthill presents a problem, the very last thing to appear are the equations, not the first! I also recommend purchasing this text, although it will not be used in the course as much as Acheson. It is however a very fine reference and a delight to read.

Feynman, R.P., Leighton, R.B., and Sands, M. 1964 *Lectures on Physics,* (Addison-Wesley: Reading, Massachusetts, USA) Vol. 2 chapters 40 and 41 deal with fluids. Lively and insightful commentary by Feynman is a great place to start learning about fluids. Section 41-2 on viscous fluids is formally in error when it discusses the stress tensor, but only for the case of a compressible fluid. The text and our course concentrate mostly on the incompressible limit. Stick with my notes, Landau and Lifschitz, or Acheson for the correct formulation of the stress tensor. The discussion itself is great physics, and if you haven't read Feynman before, you're in for a treat.

Guyon, E., Hulin, J.-P., and Petit, L. 2005, *Ce que disent les fluides,* (Belin: Paris) A beautiful book in every sense. Spectacular photographs and detailed, clear descriptions of a huge variety of fluid phenomena. Minimal use of mathematics. *En français.*

Triton, D.J. 1988, *Physical Fluid Dynamics* (Clarendon Press: Oxford) A book where the emphasis is squarely on the physics, with long discussions, many, many different applications, and very good photographs.

Standard fluid mechanics texts. These are old classics that are often referenced in the literature. The treatments are authoritative and very reliable, but as textbooks, in my opinion, these two are rather intimidating and too formal.

Landau, L.D., and Lifschitz, E.M. 1959 *Fluid Mechanics* (Pergamon Press: Oxford) Chapter on viscous fluids is concise and its presentation of the equations of motion is a very useful reference. Landau's ideas on the onset of turbulence dominated the subject until the mid-1970's, when elegant experiments showed them to be incorrect.

Batchelor, G.K. 1967, *An Introduction to Fluid Mechanics* (Cambridge University Press: Cambridge) A very long, detailed "Introduction" indeed. Good discussions of physical attributes of gases and liquids, very useful data in appendices.

Some favorite specialized texts of mine:

Tennekes, H., and Lumley, J.L. 1972, *A First Course in Turbulence* (MIT Press: Cambridge, MA) The right way to learn about turbulence. Very physical, with emphasis on formalism only where it offers the most insights.

Zel'dovich, Ya. B. and Razier, Yu 1966, *Physics of Shock Waves and High Temperature Hydrodynamical Phenomena* (Addison-Wesley: Reading, MA) Title says it all. Very readable, informative, smooth translation of Russian text. Description of atomic bomb blast waves is fascinating. (Interestingly, the Russians were allowed to quote only the American literature on the subject!)

1 Opening Comment

Displaced by the rise of quantum mechanics, hydrodynamics has all but disappeared from the curricula of physics departments. In recent years, the study of deterministic chaotic phenomena has allowed the subject to make a bit of a comeback. But most physicists receive little or no training in fluid processes.

This is a pity. From a strictly mathematical point of view, the equations in hydrodynamics are often very similar to field equations encountered in many domains of physics. It is easier to form a mental picture of a fluid than it is to imagine an abstract field. Indeed, nineteenth century physicists of the stature of Maxwell and Kelvin based their physical intuitions upon hydrodynamical analogues. This allowed them to envision the potential and solenoidal fields (as well as the wave phenomena) that emerge from studies of electrodynamics. Today, for those studying hydrodynamics, the problem is often reversed: students encountering potential fluid flow for the first time are told not to worry, it is just like electrostatics...and a vortex is just the magnetic field of a wire!

But "helpful analogy" is not the best reason for studying the physics of fluids. The best reason is that it is a rich, fascinating subject entirely in its own right. It is also, incidentally, extremely important: the problem of understanding fluid turbulence, still a very distant goal, is perhaps the only fundamental problem of modern physics which, if solved, would have immediate practical benefits. But even the problems that are reasonably well-understood are breathtaking in their scope. In this course we will talk about the weather, kitchen sinks, airplanes, sound waves, oceanic layer mixing, tsunamis, boats, hot tea, hot lava, fish, Norwegian fjords, planets, and spermatozoa (I have added a few things to the list in Acheson's book!) Surely, there must something for everyone somewhere in there. Alors, commençons.

2 Fundamentals

Although the fundamental objects of interest are the atomic particles that comprise our gas or liquid, we shall work in the limit in which the matter is regarded as a *nearly* continuous fluid. (We shall always use the term "fluid" to mean either a gas or a liquid.) The fact that this is not exactly a continuous fluid manifests itself in many ways. In a gas, the finite distance between atomic collisions makes itself felt as a form of dissipation; in a liquid, more complex small-scale interactions give rise to similar behavior. For many applications, however, such dissipation may be ignored, in which case we are treating the fluid as though it were *ideal*. We shall discuss both dissipative

and ideal flow in this course.

We start with three fundamental physical principles: mass is conserved, $F = ma$, and the first law of thermodynamics (energy is conserved). We can't go wrong with these!

2.1   Mass Conservation

Let the mass density of our gaseous fluid be given by ρ, which is a function of position vector $\boldsymbol{r}$ and time t. Consider an arbitrary volume in the fluid V. The total mass in V is

$$M = \int_V \rho \, dV \tag{1}$$

The principle of mass conservation is that the mass can't change except by a net flux of material flowing through the boundary S of the volume:

$$\frac{dM}{dt} \equiv \int_V \frac{\partial \rho}{\partial t} \, dV = -\int_S \rho \boldsymbol{v} \cdot \boldsymbol{dS}. \tag{2}$$

The divergence theorem gives

$$\int_S \rho \boldsymbol{v} \cdot \boldsymbol{dS} = \int_V \boldsymbol{\nabla}{\cdot}(\rho \boldsymbol{v}) \, dV \tag{3}$$

Since V is arbitrary, the integrands of the volume integrals must be the same, and

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}{\cdot}(\rho \boldsymbol{v}) = 0, \tag{4}$$

the differential form of the statement of mass conservation.

We can already do our first problem in hydrodynamics.

Example 1.1 Look at the water coming out of your faucet in the kitchen or bathroom. If the spray is not too hard, you should notice that the emerging stream is gently tapered, growing more narrow as it descends. What is the shape of the cross section of the stream as a function of the distance from the faucet?

The water emerging faucet is in free-fall, affected only by gravity. Let the gravitational acceleration be g, and the cross section area, a function of the distance *downward* from the faucet z, be $A(z)$. The flow is independent of time, and density of water is very nearly constant. The velocity of the water is

$$v(z)^2 = v_0^2 + 2gz \tag{5}$$

where v_0 is the emergent velocity, which follows from the conservation of energy in the constant gravitational field (or just elementary kinematics). Our time-steady equation for mass conservation is

$$\int_S \rho \boldsymbol{v} \cdot \boldsymbol{dA} = \rho[v_0 A_0 - v(z)A(z)] = 0 \tag{6}$$

where A_0 is the cross sectional area of the faucet. We thus find

$$A(z) = \frac{A_0}{\sqrt{1 + 2gz/v_0^2}} \tag{7}$$

The tapering is more pronounced when the emergent velocity v_0 is small. For a long stream, the diameter therefore contracts by a factor $\sim z^{-1/4}$.

Question: Have we made an approximation here for the velocity field? If so, what was it?

Exercise. Hold on a moment! In the example we just completed, the density was held constant, and the flow was independent of time. Doesn't that mean $\partial v/\partial z = 0$ from mass conservation? Now I'm completely confused, and you have to help. Hint: Go back and think hard about the question that was posed.

2.2 Newtonian Dynamics

2.2.1 The Lagrangian Derivative

Our second fundamental equation is a statement of Newton's second law of motion, that applied forces cause acceleration in a fluid. To make this quantitative, we introduce the idea of a fluid element. It is important to understand that this is *not* an atom! It is an intermediate quantity large enough to contain a very large number of atoms. This is a region of the fluid with all of its gross physical properties, but so small that all the macroscopic variables—density, velocity, pressure and so on—may be regarded as each having a unique value over the tiny dimensions of the element. The fluid element remains coherent enough that we may in principle follow its path through the fluid, the so-called fluid streamline.

The volume of the element dV moves with the element, and the mass of the element $\rho\, dV$ remains fixed as the element moves. Assume that the element instantaneously at position $\boldsymbol{r}$ has velocity $\boldsymbol{v}(\boldsymbol{r}, t)$. Its acceleration $\boldsymbol{a}$ is NOT

$$\frac{\partial \boldsymbol{v}}{\partial t} \tag{8}$$

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which instead would be a measure of how the velocity of the flow is changing with time at *fixed* $\boldsymbol{r}$. The acceleration must take into account the fact that the element is moving, and that the time derivative follows this motion and registers the changes of the element's properties. Therefore,

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + (\frac{d\boldsymbol{r}}{dt}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v}, \tag{9}$$

where of course $d\boldsymbol{r}/dt$ is just the velocity $\boldsymbol{v}$ of the element itself. (This is no different from the usual definition for the acceleration of, say, a planet in orbit.) We define the so-called Lagrangian derivative as so

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}, \tag{10}$$

which will prove to be a very useful operator. Hence, Newton's second law applied to our element is then

$$\rho dV \frac{D\boldsymbol{v}}{Dt} = \rho dV \left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v}\right] = \boldsymbol{F} \tag{11}$$

where the right side is the sum of the forces on the fluid element.

2.2.2 Forces acting on a fluid

The most fundamental internal force acting on a fluid is the pressure. If the fluid is a gas, the pressure is given by the ideal gas equation of state

$$P = \frac{\rho kT}{m} \equiv \rho c^2 \tag{12}$$

where T is the temperature in Kelvins, k is the Boltzmann constant 1.38×10^{-23} J K^{-1}, m is the mass per particle, and c is (for reasons we shall see later in the course) the speed of sound in an isothermal gas. The pressure is due, therefore, to the kinetic energy of the gas particles themselves.

For a liquid, matters are more simple. The density is simply a given constant, and P is solved directly as part of the problem itself, like any other flow variable.

Pressure exerts a force only if it is not spatially uniform. For example, the pressure force in the x direction on a slab of thickness dx and area $dy\,dz$ is

$$[P(x - dx/2, y, z, t) - P(x + dx/2, y, z, t)]dy\,dz = -\frac{\partial P}{\partial x}dV \tag{13}$$

There is nothing special about the x direction, so the force vector from the pressure is simply $-\boldsymbol{\nabla} P\, dV$.

Other forces can be added on as needed. We shall restrict ourselves for the moment to just one other force, of great importance in both terrestrial as well as astrophysical applications: gravity. The Newtonian gravitational acceleration $\boldsymbol{g}$ on the earth's surface is of course just a constant vector pointing (by definition) downwards. More generally, the gravitational field may be derived from a potential function

$$\boldsymbol{g} = -\boldsymbol{\nabla}\Phi. \tag{14}$$

If the field is from an external mass distribution, then Φ is a given function of $\boldsymbol{r}$ and t; otherwise Φ must be computed along with the evolution of the fluid itself. In this course, we will restrict ourselves to problems in which the gravitational potential is external, independent of the fluid properties. Combining the results of this section, we may now write down the dynamical equation of motion for a fluid subject to pressure and gravitational forces, we obtain:

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} P - \rho\boldsymbol{\nabla}\Phi \tag{15}$$

Note that the awkward volume dV has divided out, leaving us with a nice, well-posed differential equation.

2.3   Energy Equation for a Gas

The thermal energy behavior of the gas is described by the internal energy loss equation, which is most conveniently expressed in terms of the entropy per particle. Recall from thermodynamics that (up to an additive constant) this is given by

$$S = \frac{k}{\gamma - 1}\ln P\rho^{-\gamma} \tag{16}$$

where γ is the adiabatic index. At the thermodynamic level, γ is C_P/C_V, the ratio of specific heats at constant pressure and constant volume. Using methods of statistical mechanics, this can be shown to be $1 + 2/f$, where f is the number of degrees of freedom of a particle. For a monatomic gas, $\gamma = 5/3$; for a diatomic gas in which rotational degrees of freedom may be excited, $\gamma = 7/5$.

Exercise. Using simple arguments, verify that $\gamma = (f + 2)/f$.

The entropy of a fluid element is conserved unless there are explicit losses or heat sources. These could arise from radiative processes, or internal energy

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dissipation. If n is the number of particles per unit volume, then

$$nT\frac{DS}{Dt} = \frac{P}{\gamma - 1}\frac{D \ln P\rho^{-\gamma}}{Dt} = \text{net volume heating rate} \equiv \dot{Q} \qquad (17)$$

It is sometimes helpful to have an expression relating the pressure and energy densities of an ideal gas. Each degreee of freedom associated with a particle has an energy $kT/2$ in thermal equilibrium, so the energy per unit volume is $\mathcal{E} = (f/2)nkT$. Hence,

$$\mathcal{E} = \frac{P}{\gamma - 1}. \qquad (18)$$

If there are no gains, losses, or dissipation in the gas, the fluid is said to be adiabatic and the entropy of a fluid element is strictly conserved. In this case, the pressure and density are very simply related:

$$\frac{D(P\rho^{-\gamma})}{Dt} = 0. \qquad (19)$$

While the adiabatic approximation is often extremely useful, gas energetics can in general be very complicated. Each problem needs to be carefully formulated.

Exercise. Show that the combination of entropy and mass conversation implies

$$\frac{\rho}{\gamma - 1}\frac{Dc^2}{Dt} = -P\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}.$$

This is a statement of the first law of thermodynamics. Why?

2.4   Adiabatic Equations of Motion

We gather here, for ease of reference, the fundamental equations of motion for a liquid (constant density) or an adiabatic gas.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{v}) = 0 \quad \text{(Mass Conservation.)} \qquad (20)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi \quad \text{(Equation of Motion.)} \qquad (21)$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\right)\ln P\rho^{-\gamma} = 0 \quad \text{(Energy Equation.)} \qquad (22)$$

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For liquids, we use the $\gamma \to \infty$ limit of the energy equation, $D\rho/Dt = 0$. Often, this is just $\rho = \mathrm{constant}$.

It is often useful to have alternative forms of the energy equation. A mechanical energy equation is obtained by taking the dot product of $\boldsymbol{v}$ with the equation of motion. The result is

$$\frac{\rho}{2}\frac{\partial v^2}{\partial t} + (\rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\frac{v^2}{2} = -\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}P - \rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi \tag{23}$$

The left hand side of this equation may be written

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}\right) + \boldsymbol{\nabla}\boldsymbol{\cdot}\left(\frac{\rho v^2}{2}\boldsymbol{v}\right) \tag{24}$$

since the terms that make up the difference between equations (23) and (24) cancel by mass conservation. On the right side of (23), use

$$\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}P = \boldsymbol{\nabla}\boldsymbol{\cdot}(P\boldsymbol{v}) - P\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}, \tag{25}$$

and

$$\rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi = \boldsymbol{\nabla}\boldsymbol{\cdot}(\Phi\rho\boldsymbol{v}) - \Phi\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{v}) \tag{26}$$

But $\boldsymbol{\nabla}\boldsymbol{\cdot}(\rho\boldsymbol{v}) = -\partial\rho/\partial t$, and if Φ is a given function of position (as we shall assume in this course), then

$$\rho\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla}\Phi = \boldsymbol{\nabla}\boldsymbol{\cdot}(\Phi\rho\boldsymbol{v}) + \frac{\partial(\rho\Phi)}{\partial t} \tag{27}$$

Putting all of these results together gives us the energy conservation equation for a nondissipative fluid:

$$\frac{\partial}{\partial t}\left[\left(\rho\frac{v^2}{2} + \rho\Phi\right)\right] + \boldsymbol{\nabla}\boldsymbol{\cdot}\left[\left(\frac{\rho v^2}{2} + \rho\Phi + P\right)\boldsymbol{v}\right] = P\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v} \tag{28}$$

This equation applies both to liquids, in which case the right side is zero, or to adiabatic gases. In the first case, we have strict energy conservation: the time derivative of an energy density plus the divergence of the corresponding flux vanishes. For a compressible fluid, on the other hand, $P\boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{v}$ represents the expansion work done by the gas. In the exercise at the end of section 2.3, you showed that this is directly related to changes in the internal energy of the gas. Using the internal equation that you found, you should be able to show that a statement of total energy conservation follows:

$$\frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\nabla}\boldsymbol{\cdot}\boldsymbol{\mathcal{F}_{\mathcal{E}}} = 0 \tag{29}$$

where the energy density $\mathcal{E}$ is

$$\mathcal{E} = \rho \left(\frac{v^2}{2} + \Phi \right) + \frac{P}{\gamma - 1} \tag{30}$$

and the associated flux is

$$\boldsymbol{\mathcal{F_E}} = \boldsymbol{v} \left(\frac{\rho v^2}{2} + \rho \Phi + \frac{\gamma P}{\gamma - 1} \right) \tag{31}$$

In general, neither mechanical nor thermal energy is separately conserved, but their sum is conserved. Though we have not shown it, this must be true even if the gas is viscous, since dissipation does not constitute an *external* heat source. Rather, it converts mechanical into thermal energy. Only if this thermal energy is radiated away or otherwise transported across the surface of the fluid (say, by thermal conduction) is it truly lost.

3   Mathematical Matters

The study of fluids involves vector calculus manipulations that require some practice to get used to. Here we study some examples and techniques that will prove useful.

3.1   The vector "v dot grad v"

The vector $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is more complicated than it might appear. In Cartesian coordinates, matters are simple: the x component is just $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_x$, and similarly for y, z. But in cylindrical coordinates, say, the radial R component of this vector is NOT $(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R$. Rather, we must take care to write

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{\nabla}(v_R\boldsymbol{e_R} + v_\phi\boldsymbol{e_\phi} + v_z\boldsymbol{e_z}) \tag{32}$$

where the $\boldsymbol{e}$'s are unit vectors in their respective directions. In Cartesian coordinates these unit vectors are constant, but in any other coordinate system they generally change with position. Hence, *the gradient must operate on the unit vectors* as well as the individual velocity components themselves. With the help of the table that follows, your should be able to show that the radial component of $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ is

$$\boldsymbol{v}\cdot\boldsymbol{\nabla}v_R - \frac{v_\phi^2}{R}, \tag{33}$$

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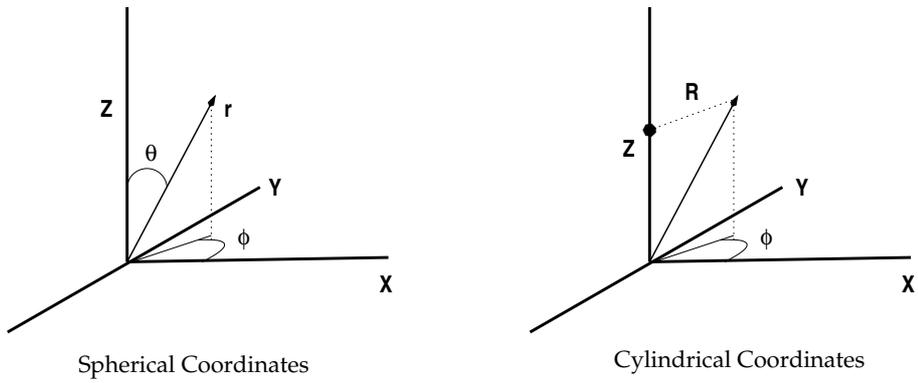
Figure 1: Spherical (r, θ, ϕ) and cylindrical coordinates (R, ϕ, z).

while the azimuthal component is

$$\boldsymbol{v} \cdot \boldsymbol{\nabla} v_\phi + \frac{v_R v_\phi}{R} \tag{34}$$

The extra terms are clearly related to centripetal and Coriolis forces, though more work is needed to extract the latter...a piece of this force still remains inside the gradient term, and emerges only when one transfers into a rotating frame.

For ease of reference, we include a table of unit vectors in cylindrical and spherical coordinates.

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Table of Unit Vectors and Their Derivatives.

Cylindrical unit vectors:

$$\boldsymbol{e_R} = (\cos\phi, \sin\phi, 0)$$
$$\boldsymbol{e_\phi} = (-\sin\phi, \cos\phi, 0)$$
$$\boldsymbol{e_z} = (0, 0, 1)$$

Spherical unit vectors:

$$\boldsymbol{e_r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$
$$\boldsymbol{e_\theta} = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$\boldsymbol{e_\phi} = (-\sin\phi, \cos\phi, 0)$$

Nonvanishing derivatives of cylindrical unit vectors:

$$\frac{\partial\boldsymbol{e_R}}{\partial\phi} = \boldsymbol{e_\phi}$$

$$\frac{\partial\boldsymbol{e_\phi}}{\partial\phi} = -\boldsymbol{e_R}$$

Nonvanishing derivatives of spherical unit vectors:

$$\frac{\partial\boldsymbol{e_r}}{\partial\theta} = \boldsymbol{e_\theta}$$

$$\frac{\partial\boldsymbol{e_r}}{\partial\phi} = \sin\theta\boldsymbol{e_\phi}$$

$$\frac{\partial\boldsymbol{e_\theta}}{\partial\theta} = -\boldsymbol{e_r}$$

$$\frac{\partial\boldsymbol{e_\theta}}{\partial\phi} = \cos\theta\boldsymbol{e_\phi}$$

$$\frac{\partial\boldsymbol{e_\phi}}{\partial\phi} = -(\sin\theta\boldsymbol{e_r} + \cos\theta\boldsymbol{e_\theta}) = -\boldsymbol{e_R}$$

3.2 Rotating Frames

It is often useful to work in a frame rotating at a constant angular velocity Ω—perhaps the frame in which an orbiting planet or a rotating fluid appears at rest. The same rule that applies to ordinary point mechanics applies here as well: add

$$-2\mathbf{\Omega} \times \mathbf{v} + R\Omega^2 \mathbf{e_R} \tag{35}$$

to the applied force (per unit mass) operating on a fluid element. As written, these rotational terms should appear on the right side of the equation of motion, with $-(1/\rho)\mathbf{\nabla}P$. The first term is the Coriolis force, the second is the centrifugal force, $\mathbf{\Omega}$ is taken to be in the vertical direction, and all velocities are measured relative to the rotating frame of reference.

Consider the following interesting case. Suppose that in a rotating frame the fluid velocity $\mathbf{v}$ is much less than $R\Omega$. Suppose further that the pressure term $(\mathbf{\nabla}P)/\rho$ is a pure gradient, either because ρ is constant, or because $P = P(\rho)$. Then the sum of the pressure, gravity, and centrifugal terms is expressible as a gradient, say $\mathbf{\nabla}H$. The steady-state fluid equation is simply

$$2\mathbf{\Omega} \times \mathbf{v} = \mathbf{\nabla}H, \tag{36}$$

since we may neglect the nonlinear $(\mathbf{v}\cdot\mathbf{\nabla})\mathbf{v}$ term in comparison with the Coriolis term. Now, since $\mathbf{\Omega}$ lies along the z axis, the left hand side of equation (36) has no z component, and therefore neither does the right hand side. That means that H is independent of z. But then the radial and azimuthal velocity components are also independent of z; that is, they are constant on cylinders! In the case of constant density, the mass conservation equation is $\mathbf{\nabla}\cdot\mathbf{v} = 0$, so that v_z is at most a linear function of z times a function of x, y, t. This function generally must vanish however, since v_z does not grow without bound at large z, and it goes smoothly to zero at a finite z boundary. Hence v_z is also z independent. The fact that small motions in rotating systems are independent of height is called the Taylor-Proudman theorem, and you will often hear people talk about "Taylor Columns" in fluid mechanics seminars. Now you know where they come from.

Another interesting application of rotating frames arises in astrophysical gas disks bound to a central mass M. The early solar system is thought to have past through such a stage. Place the origin on the central mass. We allow the disk to have a finite vertical thickness, so we pick a spot in the $z = 0$ midplane of the disk at cylindrical radius R_K. At this point, the gas orbits at the Keplerian velocity

$$v_K^2 = \frac{GM}{R_K}, \qquad \Omega_K^2 = \frac{GM}{R_K^3} \tag{37}$$

(The radial pressure gradient is assumed to be very small for these undisturbed orbits.)

We now go into a frame rotating at angular velocity Ω_K, and consider a small (x, y, z) neighborhood about midplane radius R_K: $x = R - R_K$, $y = R\phi - \Omega_K t$, and $x, y, z \ll R$. (ϕ is the azimuthal angle in cylindrical coordinates.) The sum of the radial gravitational and centrifugal forces in this neighborhood is:

$$-\frac{GM}{(R_K + x)^2 + y^2 + z^2} + (R_K + x)\Omega_K^2 \simeq 3x\frac{GM}{R_K^3} \qquad (38)$$

where only the leading order term (linear in x/R) has been retained. This residual x force is the tidal forcing from the central mass. For finite z, there is also a vertical gravitational force of $-GMz/R_K^3$, again to leading linear order in z. Assuming that external forces and pressure gradients induce only small changes in the velocity of the Keplerian orbits (generally a very good approximation), the local equation of motion for gas in a Keplerian disk is:

$$\frac{D\boldsymbol{v}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla} P + 3\Omega^2 x\boldsymbol{e_x} - \Omega^2 z\boldsymbol{e_z} + \boldsymbol{F}_{ext} \qquad (39)$$

where we have dropped the K subscript from Ω, $\boldsymbol{e_x}$ and $\boldsymbol{e_z}$ are unit vectors in the radial and axial directions, and $\boldsymbol{F}_{ext}$ represents any external forces. This equation is the starting point for understanding how the planets in the solar system interact with disk. With pressure ignored, it also used to study purely gravitational orbits. This is sometimes referred to as the Hill equation, after the astronomer who developed this approach to study the moon's orbit about the earth in the presence of the tidal field of the sun. In this case, the tidal force is due to the *sun*, Ω is the angular velocity to the earth's orbit, and the external force is the gravitational acceleration of the earth on the moon. Using these equations, numerical integration shows that if the moon were only a little farther away from the earth, the solar tidal force would have produced a highly noncircular, self-intersecting lunar orbit (as seen from the earth)! What the historical development of gravitation theory would have been under those circumstances is anybody's guess.

Another type of local approximation in a rotating frame can be used on the spherical surface of a planet or star. We use this technique in section 5.6, in which Rossby waves are discussed.

3.3   Manipulating the Fluid Equations

For a particular problem, working in cylindrical or spherical coordinates is often very convenient, but for proving general theorems or vector identities, Cartesian coordinates are usually the simplest to use. There is a formalism that allows one to work very efficiently with Cartesian fluid equations.

The index i, j, or k will each represent any of the Cartesian component x, y, or z. Hence v_i means the ith component of v, which may be any of the three (x, y, or z), depending upon which particular value of i is chosen. So v_i is really just a way to write $\boldsymbol{v}$. The gradient operator $\boldsymbol{\nabla}$ is written ∂_i, in a way that should be self-explanatory.

Next, we use the convention that if an index appears twice, it is understood that it is to be summed over all three values x, y, and z. Hence

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_i B_i, \tag{40}$$

and

$$(\boldsymbol{v}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{v} = (v_i \partial_i)v_j. \tag{41}$$

In these examples, i is a "dummy index"; in the second example the vector component of $\boldsymbol{v}$ is represented by the index j. The dynamical equation of motion in this notation is

$$\rho[\partial_t + (v_i \partial_i)]v_j = -\partial_j P - \rho \partial_j \Phi. \tag{42}$$

Using mass conservation $\partial_t \rho + \partial_i(\rho v_i) = 0$, this can also be written

$$\partial_t(\rho v_j) + \partial_i(P \delta_{ij} + \rho v_i v_j) = -\rho \partial_j \Phi, \tag{43}$$

where δ_{ij} is the Kronecker delta function (equal to unity when $i = j$, zero otherwise). The quantity

$$(P \delta_{ij} + \rho v_i v_j) \tag{44}$$

is known as the momentum flux, and in the absence of an external force, it is a conserved quantity.

Sometimes the "rot" (or "curl") operator is needed. For this, we intoduce the Levi-Civita symbol ϵ^{ijk}. It is defined as follows:

- If any of the i, j, or k are equal to one another, then $\epsilon^{ijk} = 0$.

- If $ijk = 123$, 231, or 312, the so-called even permutations of 123, then $\epsilon^{ijk} = +1$.

- If $ijk = 132$, 213, or 321, the so-called odd permutations of 123, then $\epsilon^{ijk} = -1$.

The reader can easily check that

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \epsilon^{ijk} \partial_i A_j \tag{45}$$

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Here, the vector component is represented by the index k. (Don't forget to sum over i and j!) ϵ^{ijk} is of course used in the ordinary cross product as well:

$$\boldsymbol{A} \times \boldsymbol{B} = \epsilon^{ijk} A_i B_j \tag{46}$$

Notice that

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \epsilon^{ijk} A_k B_i C_j \tag{47}$$

which proves that any even permutation of the vectors on the left side of the equation must give the same value, and an odd rearrangement gives the same value with the opposite sign.

A double cross product looks complicated:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = \epsilon^{lkm} A_l(\epsilon^{ijk} B_i C_j) = \epsilon^{mlk}\epsilon^{ijk} A_l B_i C_j. \tag{48}$$

The last equality follows because mlk is an even permutation of lkm. This looks very unpleasant, but fortunately there is an identity that saves us:

$$\epsilon^{mlk}\epsilon^{ijk} = \delta_{mi}\delta_{lj} - \delta_{mj}\delta_{li}. \tag{49}$$

The proof of this is left as an exercise for the reader, who should be convinced after trying a few simple explicit examples. With this identity, our double cross product becomes

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = B_m A_j C_j - C_m A_i B_i = \boldsymbol{B}(\boldsymbol{A} \cdot \boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B}).$$

The ijk notation also gives us a way to go from a Cartesian formulation to a vector invariant formulation in more complex situations. For example, the theory of viscosity involves the calculation of the so-called viscous tensor,

$$\sigma_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k} \tag{50}$$

The force in the j direction is proportional to $\partial_i \sigma_{ij}$. The question is how to write this tensor in terms of vector velocities and gradients in any coordinate system.

A tensor is a sort of "double vector," with two indices each behaving individually like a vector. The last term in (50) is simply a divergence, and is therefore easy to write in any coordinate system. (The delta function always behaves like a delta function in any coordinates.) The first group of derivatives does not seem to generalize quite so straightforwardly, at least

not in a way that plainly preserves its vector-like properties. For example, in cylindrical coordinates, when we calculate $\sigma_{R\phi}$, should we use

$$\frac{\partial v_R}{\partial \phi} \qquad \text{or} \qquad \boldsymbol{e_R} \cdot \left(\frac{\partial \boldsymbol{v}}{\partial \phi} \right) \ ?? \tag{51}$$

Clearly, these are not equivalent expressions.

Matters become much more clear if we write

$$\frac{\partial v_i}{\partial x_j} = [(\boldsymbol{e_j} \cdot \boldsymbol{\nabla}) \boldsymbol{v}] \cdot \boldsymbol{e_i}. \tag{52}$$

We see at once that this is obviously true in Cartesian coordinates. But the right side is just a rather elaborate vector dot product. The generalization of a dot product to any locally orthogonal coordinate system is direct and simple. Just choose whichever vector basis you would like for the $\boldsymbol{e}$ unit vectors. Although we normally reserve i, j, and k for Cartesian coordinates, the right side of equation (52) is valid, in the sense of behaving like the components of a tensor, for any choice of the $\boldsymbol{e}$ coordinate basis.

Here is another way to state what we have just written: $\partial v_i / \partial x_j$ should be thought of as the derivative of the vector $\boldsymbol{v}$ taken along the path $\boldsymbol{e_j} \cdot \boldsymbol{\nabla}$. In this sense, it is also a true vector, the directional derivative of $\boldsymbol{v}$ along $\boldsymbol{e_j}$. To find a particular component of this vector, take the dot product with $\boldsymbol{e_i}$ as above. This argument works whether the $\boldsymbol{e}$ vectors are Cartesian or not.

This is the easiest way to understand how to generalize derivative expressions of the form $\partial_i v_j$, which are not written in a nice vector invariant notation, to a vector dot product, which is coordinate independent. Practice what you have learned by demonstrating that

$$\sigma_{R\phi} = \left(\frac{1}{R} \frac{\partial v_R}{\partial \phi} + R \frac{\partial}{\partial R} \left(\frac{v_\phi}{R} \right) \right) \tag{53}$$

Does this vanish for solid body rotation? What about the divergence $\partial v_i / \partial x_i$? Does our formula give the correct expression for the divergence operator in, say, cylindrical coordinates? Once you've done that, be really ambitious and try $\epsilon^{ijk} \partial v_i / \partial x_j$. Do you get the correct expression for the curl operator? (You should.)

If you actually have done these exercises, you will have done a lot of writing! The same results can be achieved with much greater elegance using a powerful formalism known as differential geometry, in which vectors and tensors of arbitrary degrees can be handled more smoothly in a coordinate-independent manner. Differential geometry is essential in more complex situations (General Relativity, for example). But the simpler and less elegant

approach we have taken here is a better way to begin, because it is more intuitive, and in fact will suit our needs just as well.

3.4 Lagrangian Derivative of Line, Area, and Volume Elements

Often we are interested in calculating an integral over a volume of the flow, following the motion of the fluid, and calculating the change in the integral. It can be very useful to have at hand some rules about how a differential line, area, or volume element changes as it moves with the flow.

Consider a differential line element $d\boldsymbol{r}$. The difference in the velocity field $\boldsymbol{v}$ across the line element is $d\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{v}$. In a time Δt, as the line element is swept along with the flow, it will experience the following distortion:

$$d\boldsymbol{r} \rightarrow d\boldsymbol{r} + (d\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{v})\Delta t \tag{54}$$

In other words, the Lagrangian time derivative of the element $d\boldsymbol{r}$ is

$$\frac{D(d\boldsymbol{r})}{Dt} = d\boldsymbol{r}\cdot\boldsymbol{\nabla}\boldsymbol{v} \equiv d\boldsymbol{v}, \tag{55}$$

an exact differential for the velocity field.

Consider next the coordinate line elements $d\boldsymbol{x} = dx\boldsymbol{e_x}$, and the same for y and z. Each of these elements is changed in time Δt by the velocity field $\boldsymbol{v} = (v_x, v_y, v_z)$ as follows:

$$d\boldsymbol{x'} = d\boldsymbol{x} + \Delta(d\boldsymbol{x}) = (dx + [dx\partial_x v_x]\Delta t, [dx\partial_x v_y]\Delta t, [dx\partial_x v_z]\Delta t) \tag{56}$$

$$d\boldsymbol{y'} = d\boldsymbol{y} + \Delta(d\boldsymbol{y}) = ([dy\partial_y v_x]\Delta t, dy + [dy\partial_y v_y]\Delta t, [dy\partial_y v_z]\Delta t) \tag{57}$$

$$d\boldsymbol{z'} = d\boldsymbol{z} + \Delta(d\boldsymbol{z}) = ([dz\partial_z v_x]\Delta t, [dz\partial_z v_y]\Delta t, dz + [dz\partial_z v_z]\Delta t) \tag{58}$$

Thus, after time Δt, the original coordinate line elements each acquire components along all three axes. The orignal volume element is

$$(d\boldsymbol{x}\times d\boldsymbol{y})\cdot d\boldsymbol{z} = dx\,dy\,dz. \tag{59}$$

A direct calculation gives

$$(d\boldsymbol{x'}\times d\boldsymbol{y'})\cdot d\boldsymbol{z'} = dx\,dy\,dz(1 + \boldsymbol{\nabla}\cdot\boldsymbol{v}\,\Delta t) \tag{60}$$

to leading order in Δt. Hence, the Lagrangian time derivative of a volume element is

$$\frac{D(dx\,dy\,dz)}{Dt} = (dx\,dy\,dz)\boldsymbol{\nabla}\cdot\boldsymbol{v}. \tag{61}$$

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The velocity divergence is directly responsible for local volume element changes as the fluid flows.

The Lagrangian derivative of an area element dS is more tricky. We will perform the calculation by choosing a particular cubic face

$$d\boldsymbol{x} \times d\boldsymbol{y} = d\boldsymbol{S} = dS_z \boldsymbol{e_z}, \tag{62}$$

and deducing the more general vector invariant form from our specific result. A direct calculation gives to linear order in Δt:

$$d\boldsymbol{x'} \times d\boldsymbol{y'} = d\boldsymbol{x} \times d\boldsymbol{y} + dx\, dy \Delta t \left[(\partial_x v_x + \partial_y v_y)\, \boldsymbol{e_z} - \partial_y v_z \boldsymbol{e_y} - \partial_x v_z \boldsymbol{e_x} \right] \tag{63}$$

Adding and subtracting the term

$$dx\, dy\ \Delta t \partial_z v_z\ \boldsymbol{e_z}$$

on the right side of this equation turns it into something more presentable:

$$d\boldsymbol{x'} \times d\boldsymbol{y'} = d\boldsymbol{x} \times d\boldsymbol{y} + dx\, dy \Delta t\ \left[\boldsymbol{\nabla} \cdot \boldsymbol{v}\ \boldsymbol{e_z} - \boldsymbol{\nabla} v_z \right] \tag{64}$$

The z axis picks out the unique direction of the orginal surface element $d\boldsymbol{S}$, and the vector generalization of this expression is immediate and obvious:

$$d\boldsymbol{S'} = d\boldsymbol{S} + \Delta t\ (\boldsymbol{\nabla} \cdot \boldsymbol{v})\, d\boldsymbol{S} - \Delta t\ (\partial_i \boldsymbol{v}) \cdot d\boldsymbol{S} \tag{65}$$

where the notation ∂_i represents the component of the gradient operator matching the i component of $d\boldsymbol{S}$ and $d\boldsymbol{S'}$. In full index form this equation reads:

$$dS_i' = dS_i + \Delta t\ \partial_j v_j\, dS_i - \Delta t\ (\partial_i v_j) dS_j$$

The Lagrangian derivative of $d\boldsymbol{S}$ becomes

$$\frac{D(d\boldsymbol{S})}{Dt} = (\boldsymbol{\nabla} \cdot \boldsymbol{v})\, d\boldsymbol{S} - (\partial_i \boldsymbol{v}) \cdot d\boldsymbol{S} \tag{66}$$

In particular, for an arbitrary vector field $\boldsymbol{W}$,

$$\boldsymbol{W} \cdot \frac{D(d\boldsymbol{S})}{dt} = \left[(\boldsymbol{\nabla} \cdot \boldsymbol{v})\, \boldsymbol{W} - (\boldsymbol{W} \cdot \boldsymbol{\nabla}) \boldsymbol{v} \right] \cdot d\boldsymbol{S}. \tag{67}$$

3.5   The Bernoulli Equation and Conservation of Vorticity

We start with the following identity, which follows immediately from equation (49).

$$\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = \frac{1}{2} \boldsymbol{\nabla} v^2 - (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v}$$

Using this result to replace $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ in the dynamical equation of motion results in

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}v^2 - \boldsymbol{v}\times\boldsymbol{\omega} = -\frac{1}{\rho}\boldsymbol{\nabla}P - \boldsymbol{\nabla}\Phi \tag{68}$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ is known as the *vorticity*, a quantity of fundamental significance in fluid dynamics.

Let us first consider the case in which either ρ is constant, or P is a function of ρ and no other quantities. Then

$$\mathcal{H} = \int \frac{dP}{\rho},$$

the so-called *enthalpy*, is a well-defined quantity. If the pressure P is proportional to ρ^γ, then

$$\mathcal{H} = \frac{\gamma P/\rho}{\gamma - 1} \propto \rho^{\gamma - 1}. \tag{69}$$

Taking the dot product of equation (68) with $\boldsymbol{v}$ gives us

$$\frac{1}{2}\frac{\partial v^2}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\left(\frac{1}{2}v^2 + \mathcal{H} + \Phi\right) = 0 \tag{70}$$

Under steady conditions, this equation states that

$$\frac{1}{2}v^2 + \mathcal{H} + \Phi \tag{71}$$

is a constant along a streamline, a result known as *Bernoulli's theorem.* If, in addition, there is a region where the flow is uniform (at large distances for example), this constant must be the same everywhere in the flow.

This has important consequences if there is a boundary surface on which the velocity takes very different values above and below—aircraft wings, for example. Wings are designed so that the velocity is greater on the upper surface than on the lower surface. But then the constancy of $v^2/2 + H$ everywhere requires the pressure on the bottom surface of the wing to be greater than on the top. (The wing is thin, so that Φ is itself a constant!) In this way, an airplane is supported during its flight. More generally, it can be shown that the lift on a wing is directly proportional to the line integral of the velocity taken around a cross section of the wing itself. This integral is called the "circulation" Γ. The relationship between the lift force and the circulation is given by a very general relationship known as the *Kutta-Joukowski* lift theorem:

$$\text{Lift Force} = -\rho V \Gamma \tag{72}$$

where V is the velocity at large distances from the wing. The minus sign ensures that a slower velocity on the bottom generates a positive lift. We will prove this theorem later in the course, as well as derive some useful approximations for Γ.

If we take the curl of equation (68), and remember that the curl of the gradient vanishes, we find

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) = \frac{1}{\rho^2}(\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}P) \tag{73}$$

Let us once again consider the case where either ρ is constant, or when P is a function only of ρ. In that case, the right hand side vanishes and:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) = 0 \tag{74}$$

With the help of our $\epsilon^{ijk}\epsilon^{lmk}$ identity and just a little work, it is straightforward to show that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{\omega}) = 0 \tag{75}$$

is the same as

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = \frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\omega}\boldsymbol{\nabla} \cdot \boldsymbol{v} \tag{76}$$

To understand what this means, consider a closed circuit in the fluid, and perform the integral

$$\oint \boldsymbol{v} \cdot \boldsymbol{dr} \equiv \int \boldsymbol{\omega} \cdot \boldsymbol{dS} \tag{77}$$

where in the integral on the right, the area is bounded by the original circuit. The integral is just the circulation we discussed in the previous section, and it is conserved as it moves with the flow. More generally, one speaks of *vorticity conservation:* vorticity is conserved, in the sense that the vorticity flux through an area moving with fluid does not change.

This is now simple to prove, because we have already done all the hard work. Moving with the flow,

$$\frac{D}{Dt}(\boldsymbol{\omega} \cdot \boldsymbol{dS}) = \left(\frac{D\boldsymbol{\omega}}{Dt}\right) \cdot \boldsymbol{dS} + \boldsymbol{\omega} \cdot \frac{D(\boldsymbol{dS})}{Dt}. \tag{78}$$

Using (67) and (76), one sees immediately that this adds up to zero! Later in the course, we will give another proof of this important theorem, without using an area integral.

We may simplify (76) somewhat. Mass conservation implies

$$\frac{D \ln \rho}{Dt} = -\boldsymbol{\nabla} \cdot \boldsymbol{v} \tag{79}$$

so that our $\boldsymbol{\omega}$ equation becomes

$$\frac{D\boldsymbol{\omega}}{Dt} - \boldsymbol{\omega}\frac{D \ln \rho}{Dt} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v}, \tag{80}$$

or

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{1}{\rho}(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{81}$$

In strictly two-dimensional flow, this is a very powerful constraint. Then $\boldsymbol{\omega}$ has only a z component, and the right side must vanish. Under these circumstances, we work with equations that have been integrated over the vertical direction, and use the surface density $\Sigma = \int \rho dz$. We then obtain:

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\Sigma}\right) = 0 \tag{82}$$

This is known as the conservation of potential vorticity. It is extremely useful in the study of two-dimensional turbulence, and in studying wave propagation in planetary atmospheres.

Exercise. Consider time-independent rotational flow, with $v_\phi(R, z) = R\Omega(R, z)$ in cylindrical coordinates. All other velocity components vanish. Assume that vorticity conservation holds. Prove that Ω cannot, in fact, depend upon z!

Exercise. Ertel's theorem. In equation (81), if we take the dot product with the entropy gradient $\boldsymbol{\nabla}S$, we obtain

$$\boldsymbol{\nabla}S \cdot \frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \frac{\boldsymbol{\nabla}S}{\rho} \cdot (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v} \tag{83}$$

You may think that if an entropy gradient is present, then we in should in general retain the term in equation (73) proportional to $\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}P$. But if the entropy can be written as a function of the thermodynamic variables P and ρ ($S = S(P, \rho)$), as is very often the case, this cross term vanishes when dotted with entropy gradient. Why?

Prove that the above equation may be written in the form

$$\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla}S\right) = \frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla}\left(\frac{DS}{Dt}\right). \tag{84}$$

If the motions are such that the Lagrangian change in S is very small, then the left side of the equation may be ignored, and

$$\frac{\boldsymbol{\omega}}{\rho} \cdot \boldsymbol{\nabla} S$$

is itself conserved with the motion of a fluid element. Therefore, even if P is not a function of ρ alone and vorticity is generated by $\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} P$ torques, this dot product *is* still conserved. This is known as Ertel's theorem, and it is used all the time by geophysicists studying the atmosphere and the oceans.

Exercise. Conservation of helicity. The helicity of a region of fluid is defined to be

$$\mathcal{H} = \int \boldsymbol{\omega} \cdot \boldsymbol{v} \, dV$$

where the volume integral is taken over the fluid region. Assume that $\boldsymbol{\omega} \cdot \boldsymbol{n}$ vanishes when integrated over the surface bounding the region, where $\boldsymbol{n}$ is the unit normal to the area, and that the conditions for Kelvin's circulation theorem hold. Prove that the helicity $\mathcal{H}$ is conserved moving with the fluid:

$$\frac{D\mathcal{H}}{Dt} = 0$$

Do *not* assume that the flow is incompressible.

3.6   Solutions of the Laplace Equation

Consider an incompressible flow described by $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$. If the flow is also irrotational, then $\boldsymbol{v}$ may be derived from a gradient, $\boldsymbol{v} = \boldsymbol{\nabla}\Psi$. These two equations imply

$$\nabla^2 \Psi = 0, \tag{85}$$

which is the equation of Laplace. Note that this is true even if the the curl of $\boldsymbol{v}$ is finite: any vector field can be expressed as the sum of the gradient of a scalar potential plus the curl of a vector potential. The scalar potential of the velocity field must always satisfy the Laplace equation if the flow is incompressible; information about the vector potential is lost when the divergence of $\boldsymbol{v}$ is taken.

The Laplace equation arises often in fluid mechanics. We have just seen one simple example, but there are many others. The gravitational potential satisfies the Laplace equation for example (when its sources are external to the fluid), and the pressure very nearly satisfies the Laplace equation in a highly viscous, steady flow. It is of interest to familiarize ourselves with some of its simple solutions.

It is possible to get quite far by using symmetry arguments, and by finding solutions in one coordinate system (where they are obvious) and writing them in other coordinates (where they are not so obvious). For example, in Cartesian coordinates:

$$\nabla^2 \Psi = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \Psi = 0 \tag{86}$$

So, *obviously,* three solutions are Ψ equals x, y, or z. In spherical coordinates, however,

$$\nabla^2 \Psi = \frac{1}{r} \frac{\partial^2 (r\Psi)}{\partial r^2} + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \Psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} = 0 \tag{87}$$

We have just shown that three solutions of this equations are Ψ equals $r\sin\theta\cos\phi$, $r\sin\theta\sin\phi$, and $r\cos\theta$. That, at least on pure mathematical grounds, is not immediately obvious. But you can plug in our solutions and satisfy yourself that they really are valid.

Conversely, an obvious solution in spherical coordinates is $\Psi = 1/r$. It is not so obvious that

$$\Psi = \left(x^2 + y^2 + z^2 \right)^{-1/2} \tag{88}$$

satisfies the Cartesian Laplace equation, but it must, and it does.

The spherical Laplace equation has picked out one point in space to be the origin, $r = 0$. *Obviously,* this could be any point, and the "translation symmetry" of the Laplace operator is most apparent in Cartesian coordinates. Clearly, if $\Psi(x, y, z)$ is a solution of the Laplace equation, then so is

$$\Psi' \equiv \Psi(x - x', y - y', z - z') \tag{89}$$

where $\boldsymbol{r'} = (x', y', z')$ is an arbitrary constant vector. Thus, if $1/r$ is our point source solution, then so must be

$$\Psi = \frac{1}{|\boldsymbol{r} - \boldsymbol{r'}|} \tag{90}$$

The symmetry here is obvious physically, but not mathematically! Indeed, it is such a powerful mathematical constraint that we shall now generate *all* the axisymmetric solutions to the spherical Laplace equation from this one—obvious?—solution.

Consider the special case in which $\boldsymbol{r'}$ lies along the z axis. Then

$$|\boldsymbol{r} - \boldsymbol{r'}|^{-1} = \left(r^2 + (r')^2 - 2rr' \cos\theta \right)^{-1/2} \tag{91}$$

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Let r become arbitrarily large. Then

$$|\boldsymbol{r} - \boldsymbol{r'}|^{-1} = \frac{1}{r} \left(1 - 2(r'/r)\cos\theta + (r'/r)^2\right)^{-1/2}, \tag{92}$$

and we may expand the right side in a power series in the small quantity (r'/r). We then find

$$|\boldsymbol{r} - \boldsymbol{r'}|^{-1} = \frac{1}{r} + \frac{r'\cos\theta}{r^2} + \left(\frac{r'^2}{r^3}\right)\frac{3\cos^2\theta - 1}{2} + \ldots \tag{93}$$

The nth term in the series will be r'^n/r^{n+1} times a polynomial of degree n in $\cos\theta$, the so-called *Legendre* polynomials $P_n(\cos\theta)$:

$$|\boldsymbol{r} - \boldsymbol{r'}|^{-1} = \sum_{n=0}^{\infty} \left(\frac{r'^n}{r^{n+1}}\right) P_n(\cos\theta) \tag{94}$$

Notice, however, that our choice of letting r become much greater than r' was entirely arbitrary. We could have equally well let r' become large. Under those conditions, we would have the above solution with r and r' reversed. Therefore, in general,

$$|\boldsymbol{r} - \boldsymbol{r'}|^{-1} = \sum_{n=0}^{\infty} \left(\frac{r_<^n}{r_>^{n+1}}\right) P_n(\cos\theta) \tag{95}$$

where $r_<$ $(r_>)$ is the smaller (larger) of r and r'.

The sum on the right hand side of the equation is a solution to the Laplace equation in spherical coordinates. But it is also a superposition of functions that are power laws in r times a polynomial in $\cos\theta$. Since the Laplace equation is linear and homogeneous in r (each term scales as $1/r^2$), it follows that the individual terms in the sum must each *separately* satisfy the Laplace equation. Thus, we have found an infinite number of solutions. Moreover, it can be shown that the $P_n(\cos\theta)$ functions form a complete basis, so that we have found all of the axisymmetric solutions of the Laplace equation that are necessary.

In fact, for our purposes in this course, we will not need an infinite number of solutions. The most useful to us will be those proportional to P_0, P_1, and P_2:

$$(A + B/r), \qquad (Ar + B/r^2)\cos\theta, \qquad (Ar^2 + B/r^3)(3\cos^2\theta - 1)/2, \quad (96)$$

where A and B are constants. Notice that replacing $\cos\theta$ by either $\sin\theta\cos\phi$ or $\sin\theta\sin\phi$ still gives valid solutions. That is "obvious." Why?

3.7 Gravitational Tidal Forces

As an illustration of how the expansion of the potential function can be used, let us calculate the height of the tides that are raised on the earth by the moon. (Our calculation will actually be completely general for any two body problems, apart from the specific numbers we use.) The oceans of the earth form surfaces satisfying the equation of hydrostatic equilibrium,

$$\boldsymbol{\nabla} P = -\rho \boldsymbol{\nabla} \Phi. \tag{97}$$

The interface between sea and air is an equipotential surface. We are interested in how these surfaces differ from spheres when the presence of the moon is taken into account.

We define the z axis to be along the line joining the centers of the earth and the moon. The distance between the earth and moon centers will be r, and a point on the earth's surface is at a vector location $\boldsymbol{r} + \boldsymbol{s}$ relative to the center of the moon. Let $\boldsymbol{s} = (x, y, z)$ in Cartesian coordinates with origin at the center of the earth. Note that

$$\frac{1}{|\boldsymbol{r} + \boldsymbol{s}|} = (r^2 + s^2 + 2rs\cos\theta)^{-1/2}, \tag{98}$$

where θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{s}$. We need to keep track of the sign, which is different from our r, r' expansion. We regard r as fixed, and calculate forces by taking gradient with respect to x, y, and z. We have

$$-\frac{GM_m}{|\boldsymbol{r} + \boldsymbol{s}|} = -\frac{GM_m}{r}\left[1 - \frac{sP_1(\cos\theta)}{r} + \left(\frac{s}{r}\right)^2 P_2(\cos\theta) + ...\right] \tag{99}$$

where M_m is the mass of the moon. Differentiating with respect to $z = s\cos\theta$ gives, to first approximation

$$-\frac{\partial \Phi}{\partial z} = -\frac{GM_m}{r^2} \tag{100}$$

which looks familiar: it is the Newtonian force acting between the centers of the two bodies, directing along the line joining them. The direct force is cancelled by a centrifugal force in the frame of the earth–moon orbit. The tidal force comes in at the next level of approximation. The tidal potential is:

$$\Phi \text{ (tidal)} = -\frac{GM_m s^2}{r^3}P_2(\cos\theta) = -\frac{GM_m}{r^3}[z^2 - \frac{(x^2+y^2)}{2}] \tag{101}$$

The tidal force, after carrying out the gradient operation, is

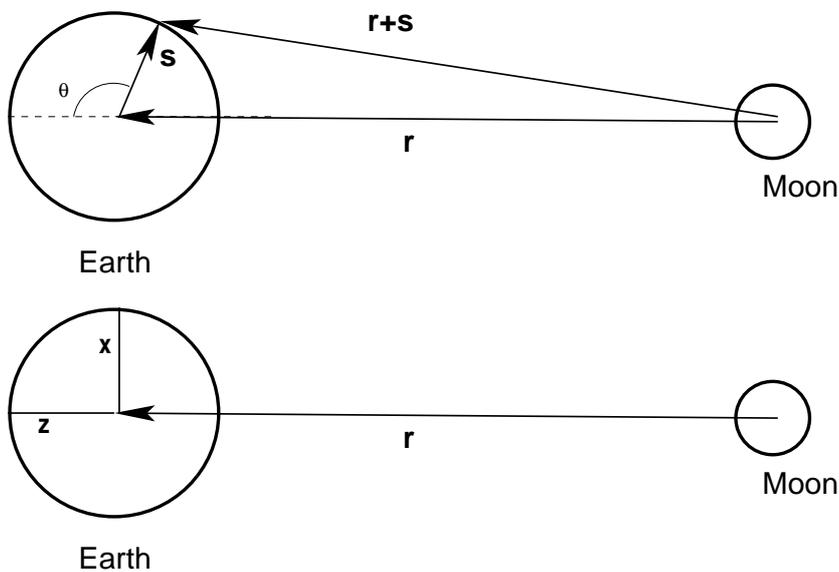$$\boldsymbol{g} \text{ (tidal)} = -\boldsymbol{\nabla}\Phi = \frac{GM_m}{r^3}(-x, -y, 2z) \tag{102}$$

Figure 2: Above: the vectors **r** and **s** and angle θ used in calculating the tides raised on the earth by the moon. Below: Cartesian coordinates centered on the earth's core. x and z are shown; y points into the page.

Tidal forces squeeze inward in the directions perpendicular to the line joining the bodies, and stretch along the direction defined by this line.

Note that we have calculated only the forces: the sea displacements that result from these forces can be very complicated! The displacements are not necessarily in phase with the tidal force, and temporal oscillations together with local conditions can produce exceptionally large tides (e.g., the Bay of Fundy), or very small tides (e.g., the Mediterranean Sea).

Let us assume, however, that the new shape of the earth has adjusted so that the surface is an equipotential of the combined gravitational fields of the earth plus the moon. Let Φ_1 be the potential function of the earth's unperturbed spherical field, $-GM_e/s$. Let Φ_2 be the changed potential function in the presence of the moon's potential; Φ_2 differs only slightly from Φ_1. More precisely, the presence of the moon introduces two terms: i.) the direct gravitational potential $GM_m z/r^2$, whose gradient force is exactly canceled by the centrifugal force, plus ii.) a term proportional to $P_2(\cos\theta)$. It is this P_2 potential that is the leading order term we should retain. (The centrifugal potential also introduces a tidal term at this order, but here we will ignore this effect.) The lunar tidal potential causes a displacement of all the original spherical equipotential surfaces, and it is this quantity $\boldsymbol{\xi}$ that we wish to calculate. Imagine following the distortion of one particular *fixed*

value equipotential surface as the moon's influence is added. Then by explicit assumption, Φ_2 has the same value as the original equipotential surface Φ_1, only now at another location: after the surface has been displaced by $\boldsymbol{\xi}$. Thus,

$$\Phi_2(\boldsymbol{s} + \boldsymbol{\xi}) = \Phi_2(\boldsymbol{s}) + \boldsymbol{\xi} \cdot \boldsymbol{\nabla}\Phi_2 = \Phi_1(\boldsymbol{s}) \tag{103}$$

where s is the radius of the earth. But

$$\Phi_2(\boldsymbol{s}) - \Phi_1(\boldsymbol{s}) \equiv \Phi \text{ (tidal)} \tag{104}$$

and to leading order we may replace Φ_2 with Φ_1 in the $\boldsymbol{\xi} \cdot \boldsymbol{\nabla}$ term. Then

$$\Phi \text{ (tidal)} = -\frac{GM_m s^2}{r^3}P_2(\cos\theta) = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\Phi_1 = -\xi_s\frac{GM_e}{s^2} \tag{105}$$

or

$$\xi_s = s\frac{M_m}{M_e}\left(\frac{s}{r}\right)^3 P_2(\cos\theta) \tag{106}$$

This works out to be

$$\xi_s = 0.32 P_2(\cos\theta) \text{ meters} \tag{107}$$

for the earth-moon system. The sun's effect is about one-third as large, and depending on the lunar phase, can either enhance or offset the moon's tidal force. (There is also the neglected centrifugal tide, of comparable magnitude.) The biggest tidal enhancement occurs at either new moon or full moon. (Why?)

Notice how extremely sensitive the height of the tidal displacement is to the separation distance r. When the moon was closer to the earth a billion years ago, as it is believed to have been, the tidal displacements were almost an order of magnitude larger.

Exercise. The equation

$$\Phi \text{ (tidal)} = -\boldsymbol{\xi} \cdot \boldsymbol{\nabla}\Phi_1$$

has a simple mechanical interpretation in terms of "work done" and "potential energy." What is it?

4   Waves

Small disturbances in fluids propagate as waves. Since quantum mechanics ascribes wave behavior even to ordinary particles, almost everything in

physics seems to be some kind of wave or another. Wave propagation in fluids is fascinating and remarkably subtle. The study of waves has been an important stimulant for the development of mathematics. For example, the entire field of spectral methods (expressing a complicated function as a linear superposition of simple functions) grew from Fourier's attempts to represent general disturbances as a superposition of waves. Finally, the study of waves can reveal far-reaching properties of the fluid equations that other types of solution cannot. This is because when we study small amplitude waves, we can often obtain rigorous analytic, time-dependent solutions without any assumptions of spatial symmetry. Normally, such analytic results must be time-steady and/or highly symmetric.

In this section, we will study the properties of waves in a great variety of different systems.

4.1   Small Perturbations

Waves are said to be linear or nonlinear according to whether their associated amplitudes are much smaller than, or in excess of or comparable to, the corresponding equilibrium values of the background medium. For example, if at a particular point in a fluid the equilibrium pressure is $P(\boldsymbol{r})$, and a wave disturbance at time t causes the pressure to change to $P'(\boldsymbol{r}, t)$, then in linear theory,

$$P'(\boldsymbol{r}, t) - P(\boldsymbol{r}) \equiv \delta P \ll P(\boldsymbol{r}) \tag{108}$$

For the velocity, linear theory generally requires the disturbance to be much less than $\sqrt{P/\rho}$, *not* the velocity of the background. The flow velocity itself is irrelevant, since relative motion by itself does not affect local physics! (Velocity *gradients* in the equilibrium flow are a different matter, however. They can, in fact, be critical for understanding wave propagation.) The name "linear" refers to the fact that in the mathematical analysis, only terms *linear* in the δ amplitudes are retained, while terms of quadratic or higher order are ignored.

Small disturbances can be described mathematically in more than one way. The above equation for δP is known as an Eulerian perturbation, which is the difference between the equilibrium and perturbed values of a fluid quantity taken *at a fixed point in space.* It is sometimes useful to work with what is known as a Langrangian perturbation, particularly when freely moving boundary surfaces are present. In a Lagrangian disturbance, we focus not upon the change at a fixed location $\boldsymbol{r}$, but upon the changes associated with a particular *fluid element* when it undergoes a displacement $\boldsymbol{\xi}$. For the case of a pressure disturbance, for example, we ask ourselves how does the pressure of a fluid element change when it is displaced from its equilibrium

value $\boldsymbol{r}$ to $\boldsymbol{r} + \boldsymbol{\xi}$? The Langrangian perturbation ΔP is therefore

$$P'(\boldsymbol{r} + \boldsymbol{\xi}, t) - P(\boldsymbol{r}) \equiv \Delta P. \tag{109}$$

Note the difference between equations (108) and (109). To linear order in ξ, ΔP and δP are related by

$$\Delta P = P'(\boldsymbol{r}, t) - P(\boldsymbol{r}) + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} P = \delta P + \boldsymbol{\xi} \cdot \boldsymbol{\nabla} P. \tag{110}$$

The Lagrangian velocity perturbation $\Delta \boldsymbol{v}$ is given by $D\boldsymbol{\xi}/Dt$:

$$\Delta \boldsymbol{v} \equiv \frac{D\boldsymbol{\xi}}{Dt} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} \tag{111}$$

where $\boldsymbol{v}$ is any background velocity that is present. This is simply the instantaneous time rate of change of the displacement of a fluid element, taken relative to the unperturbed flow. Since

$$\Delta \boldsymbol{v} = \delta \boldsymbol{v} + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{v}, \tag{112}$$

the Eulerian velocity perturbation $\delta \boldsymbol{v}$ is related to the fluid displacement $\boldsymbol{\xi}$ by:

$$\delta \boldsymbol{v} = \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \tag{113}$$

Exercise. Let $\boldsymbol{v} = R\Omega(R)\boldsymbol{e}_\phi$. Consider a displacement $\boldsymbol{\xi}$ with radial and azimuthal components ξ_R and ξ_ϕ, each depending upon R and ϕ. Show that

$$\frac{D\xi_R}{Dt} = \delta v_R, \quad \frac{D\xi_\phi}{Dt} = \delta v_\phi + \xi_R \frac{d\Omega}{d\ln R}$$

where $D/Dt = \partial/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla}$. (Be careful!)

Exercise. Compare our expression for $\delta \boldsymbol{v}$ with equation (76) for vorticity conservation. For the case in which $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$, show that $\delta \boldsymbol{v}$ is nonvanishing only if $\boldsymbol{\xi}$ is NOT frozen into the flow (like vorticity!). Why should the restriction $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ be important?

We may think of δ and Δ as difference operators, something like ordinary differention. For example,

$$\delta \left(\frac{1}{\rho}\right) = \frac{1}{\rho + \delta\rho} - \frac{1}{\rho} = -\frac{\delta\rho}{\rho^2}, \tag{114}$$

since we work only to linear order. But care must be taken when partial derivatives are present! Note that, for example,

$$\delta \frac{\partial P}{\partial x} = \frac{\partial(\delta P)}{\partial x} \tag{115}$$

BUT:

$$\Delta \frac{\partial P}{\partial x} = \frac{\partial \Delta P}{\partial x} - \frac{\partial \boldsymbol{\xi}}{\partial x} \cdot \boldsymbol{\nabla} P \neq \frac{\partial \Delta P}{\partial x} \tag{116}$$

In other words, δ commutes with the ordinary Eulerian partial derivatives with respect to time and space, but Δ does not. It is also possible to have precisely zero Eulerian pertubations, and yet have finite Lagrangian displacements and perturbations! In this case, there are no physical disturbances at all: instead, by displacing fluid elements and giving them exactly the undisturbed value at the new location, we have simply relabeled the coordinates without disturbing the fluid. In a given problem, when we use Lagrangian disturbances, we must take care to ensure that true physical disturbances are being calcuated. In this sense, Eulerian disturbances are less prone to misunderstanding. An Eulerian perturbation is always a real physical change!

To understand better the motivation for defining a Lagrangian perturbation, consider waves on the surface of the sea. The pressure at the air-sea interface remains fixed (essentially zero) as the wave passes. If $z = 0$ is the unperturbed surface, the boundary condition satisfied by the wave is not $\delta P(0) = 0$, since the pressure at the location of the unperturbed surface in fact changes. Instead, the boundary condition is $\Delta P(0) = 0$, since this constant pressure condition "moves" with the displaced fluid element. This may be written:

$$\delta P = -\xi \frac{\partial P}{\partial z} \tag{117}$$

where all quantities are evaluated at the unpertubed $z = 0$ surface. In simple circumstances, $\partial P/\partial z = -\rho g$ where g is the downward acceleration of gravity, and thus $\delta P = \rho g \xi$. Can you give a simple physical interpretation of this equation?

Exercise. Go back to the section on tidal forces, and give an interpretation of our analysis in terms of the Eulerian perturbation $\delta \Phi$ and the Lagrangian perturbation $\Delta \Phi$. What corresponds to a "fluid element" in this problem? Note that this is an example of a vanishing Lagrangian perturbation with a finite Eulerian perturbation.

4.2 Water Waves

It is time to get wet.

Consider a body of water of depth H in a constant downward pointing gravitational field $-g$. In equilibrium there is no velocity and the pressure is given by $P = -\rho g z$ ($z < 0$ in the water). The fundamental linear equations of motion for small disturbances are

$$\boldsymbol{\nabla}{\cdot}\delta\boldsymbol{v} = 0, \tag{118}$$

$$\frac{\partial \delta v_z}{\partial t} = -\frac{1}{\rho}\frac{\partial \delta P}{\partial z} \tag{119}$$

$$\frac{\partial \delta v_x}{\partial t} = -\frac{1}{\rho}\frac{\partial \delta P}{\partial x} \tag{120}$$

Notice that the gravitational field does not appear in the linear equations themselves, since it has no Eulerian perturbation. (It is what it is.) x is a direction perpendicular to z. If we take $\partial/\partial z$ of equation (119) and add $\partial/\partial x$ of equation (120) we find that the pressure satisfies the Laplace equation:

$$\frac{\partial^2 \delta P}{\partial x^2} + \frac{\partial^2 \delta P}{\partial z^2} = 0 \tag{121}$$

This is a bit unexpected for a wave equation! Let us assume that all small δ quantities are proportional to $\exp(ikx - i\omega t)$ times a function of z. As usual, we take only the real part when the true physical quantity is needed. The amplitudes δP and $\delta \boldsymbol{v}$ could in principle be complex numbers. This form is permitted because the coefficients in the linearized equations do not depend upon any spatial variables or time. k is known as the wavenumber, and ω is known as the angular frequency. Let λ be the wavelength and T the period of this sinusoidal wave. Then $k = 2\pi/\lambda$, and $\omega = 2\pi/T$.

Taking δP to be now the z-dependent amplitude of the perturbed pressure, we find that it must satisfy the differential equation

$$\frac{d^2 \delta P}{dz^2} - k^2 \delta P = 0 \tag{122}$$

Let us take $k > 0$ without any loss of physical generality. (The sign of k depends upon which direction we choose to be the positive x direction along the water's surface.) Then our solutions are e^{kz} and e^{-kz}. If

$$\delta P = \exp(kz) + A\exp(-kz) \tag{123}$$

with A an integration to be chosen, then equation (119) gives

$$\delta v_z = -\frac{ik}{\rho\omega}(\exp(kz) - A\exp(-kz)) \tag{124}$$

Since the equilibrium state has no velocity, the vertical displacement ξ_z follows from $\delta v_z = -i\omega\xi_z$, or

$$\xi_z = \frac{i\delta v_z}{\omega} = \frac{k}{\rho\omega^2}(\exp(kz) - A\exp(-kz)) \tag{125}$$

At the bottom of the sea $z = -H$, $\delta v_z = 0$. Hence $A = \exp(-2kH)$. This implies

$$\xi_z = \frac{2ke^{-kH}}{\rho\omega^2}\sinh[k(z+H)] \tag{126}$$

and

$$\delta P = 2e^{-kH}\cosh[k(z+H)] \tag{127}$$

where sinh and cosh are the usual hyperbolic sine and cosine functions. To relate ω to k, we use the free surface pressure boundary condition at $z = 0$:

$$0 = \delta P + \xi_z\frac{\partial P}{\partial z} = 2e^{-kH}[\cosh(kH) - \frac{gk}{\omega^2}\sinh(kH)] \tag{128}$$

which becomes

$$\omega^2 = gk\tanh(kH) \tag{129}$$

This relationship between the wave angular frequency ω and wavenumber k is known as a *dispersion relation,* and it is the most important equation in determining the qualitative behavior of linear waves. The relation

$$\text{frequency} \times \text{wavelength} = \text{velocity} \tag{130}$$

that one learns in one's first physics course is equivalent to the dispersion relation

$$\omega^2 = k^2c^2 \tag{131}$$

where $c = \omega/k$ is the "wave velocity." For simple sound waves (see section 5.3) or for light waves, c is a constant independent of the wavenumber. In general, however, the velocity c *does* depend upon the wavenumber. For water waves, equation (129) gives

$$c^2(\text{water waves}) = gH\left(\frac{\tanh kH}{kH}\right) \tag{132}$$

For very long wavelengths $kH \ll 1$ (so very shallow seas also work), $c^2 = gH$, and these so-called long waves have a velocity independent of k, like light or sound. Earthquakes can sometimes generate such long wavelength disturbances. They are known as "tsunamis." The fact that different wavenumbers travel at the same velocity means that a strong wave pulse will retain its

form as it travels. By contrast, short wavelength (or deep water) waves have $c^2 = g/k$. This means that smaller wavelength disturbances move slower, and a wave pulse composed of many different individual wavenumbers spreads out with time. In deep water, longer wavelength components race ahead of smaller wavelength components. After a relatively short time, the initial pulse has spread out, and has only a small peak amplitude. By contrast, a long wavelength tsunami coming into shallow coastal seas retains the large, unspread amplitude that it had in the open sea. When the raised sea bottom makes itself felt, the dispersion relation $\omega^2 = gHk^2$ demands that a wave of a given frequency has a *growing* wavenumber as the sea depth H gets smaller. The waves "pile up" as the distance between successive wave crests decreases, and the velocity of these wave crests $\omega/k = \sqrt{gH}$ becomes smaller with diminishing H. The wave inevitably grows in amplitude, simply to conserve its energy! Higher elevations *within the wave* begin to move faster than lower elevations in this truly nonlinear disturbance, and the wave breaks. If the amplitude is not too large, it is just another fun day at the beach. But if the wave is a tsunami, the results can be disastrous in more than just a mathematical sense.

4.2.1 Hydraulic Jumps

A slightly less spectacular but no less interesting nonlinear phenomenon related to surface waves on water is known as the *hydraulic jump.* If you turn on a water faucet and allow a strong stream to strike the bottom of the sink, you will see the following behavior. Near the incoming stream, the water is relatively shallow and moving rapidly. Then, quite abruptly, the height of the water jumps and the flow slows as it moves to the edge of the sink. Why is there a sudden jump in height?

What you are observing is something like a shock wave in the water. The wave speed $c = \sqrt{gH}$ represents the rate at which signals—including causal behavior—are propagated in shallow water. The water emerging from the faucet and spreading in the sink is traveling faster than c (about 20 cm s^{-1}) near the contact point where the incoming stream strikes. The water does not "know" that the sink has a wall, because no signal can travel upstream against this velocity. A sort of transition occurs some at some stand-off distance from the wall. The depth of the water increases, the velocity decreases, and signals may then propagate through the slower-moving liquid, allowing it to adjust to the presence of the wall. This rapid change is called a hydraulic jump.

The presence of a wall is by no means necessary for the occurence of a hydraulic jump. Tidal changes in rivers and estuaries can induce velocities in excess of c. In this context, hydaulic jumps are called "bores." There are

many well-known bores around the world, including France. They can be sufficiently vigorous that people sometimes are able to surf on river bores! There is a famous bore on *la Dordogne* known simply as le Mascaret ("the bore," for anglophiles). Parisians of a certain age will recall that *la Seine* used to have its own locally-named bore, la Barre. It disappeared in the 1960's, eliminated by dredging activities that changed the shape of the river bottom.

Consider a one-dimensional flow of shallow water in the $+x$ direction. We denote the density as ρ, the velocity U_1 and the height H_1. The pressure P_1 is a function of height z: $P_1 = \rho g (H_1 - z)$. (The velocity U_1 is nearly independent of z for shallow water.) We will work with the height-integrated equations of motion, for which we shall need the result

$$\int_0^{H_1} P_1 dz = \frac{1}{2} \rho g H_1^2.$$

The fluid makes a transition at some point from state 1 to state 2, with corresponding flow variables U_2, H_2, and P_2. In the transition region, there will be motion both in the x and z directions, but if we begin by integrating over z, we are left with a one-dimensional problem in x. Thus, mass conservation becomes

$$U_1 H_1 = U_2 H_2 \tag{133}$$

while momentum conservation (essentially a balance between the effective pressure forces) becomes:

$$U_1^2 H_1 + \frac{g H_1^2}{2} = U_2^2 H_2 + \frac{g H_2^2}{2}. \tag{134}$$

Eliminating U_2 from the equations gives

$$\frac{U_1^2}{g H_1} = \frac{H_2 (H_1 + H_2)}{2 H_1^2} \tag{135}$$

and by symmetry

$$\frac{U_2^2}{g H_2} = \frac{H_1 (H_1 + H_2)}{2 H_2^2} \tag{136}$$

The ratio $F = U^2/gH$ is called the *Froude number*, and it plays a role analogous to the *Mach* number (as we shall see) for sound waves. The hydraulic jump gets its name of course because it really is a jump: $H_2 > H_1$. Then $F_1 > 1$, while $F_2 < 1$. The fluid enters "supercritical" and exits "subcritical." Indeed, if we now solve equation (135) for H_2, we find the one physical solution is

$$\frac{H_2}{H_1} = \frac{\sqrt{1 + 8 F_1} - 1}{2}, \tag{137}$$

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an explicit solution for H_2 (and thus $U_2 = H_1 U_1 / H_2$) in terms of the upstream 1-variables.

Notice that our solution H_2/H_1 does give solutions for $F_1 < 1$ that lead to to $H_2 < H_1$, but these correspond to a jump in the $2 \to 1$ direction. The "arrow of time," in the sense of whether the 2-side or the 1-side is the initial condition, comes from energy dissipation considerations, and is a separate piece of physics. The hydraulic jump does not conserve mechanical energy, it dissipates it as heat. The second law of the thermodynamics tells us that mechanical energy can spontaneously dissipate into heat, but never the reverse.

Start with the energy flux for an incompressible flow:

$$U[\frac{\rho U^2}{2} + P + \rho g z] \tag{138}$$

where the final term is the potential energy. Substituting for the pressure P and integrating over height gives

$$\mathcal{E} \equiv \rho \left(\frac{H U^3}{2} + g U H^2 \right) \tag{139}$$

Using equations (133), (135), and (136) leads to

$$\mathcal{E}_2 - \mathcal{E}_1 = \frac{\rho g U_1}{4 H_2} (H_1 - H_2)^3. \tag{140}$$

This is the rate at which energy is being dissipated in the jump. It is proportional to $(H_1 - H_2)^3$, which means energy is lost if and only if $H_1 < H_2$. This, ultimately, is why only hydraulic jumps are observed in nature, and never a "hydraulic fall."

4.2.2 Capillary Phenomena

At sufficiently small wavelengths, gravity is no longer the dominant restoring force for surface water waves. The fact that water has a surface tension, and acts like a membrane, must be taken into account.

Physically, surface tension arises because water molecules attract one another weakly by electrical dipole-dipole interactions. There is an attractive force at the surface-air interface, where molecules are pulled down from neighbors below but not up from the air above. The measured dipole moment of a water molecule is $p = 6.2 \times 10^{-30}$ C-m ("Coulomb-meters"). The density of

water is 10^3 kg m^{-3}, and each H$_2$O molecule has a mass of 18 proton masses. This gives an average inter-molecular separation of about $r = 3 \times 10^{-10}$ m, and typical dipole field of $E = p/(4\pi\epsilon_0 r^3)$. We expect, therefore, that a characteristic potential energy per unit area (these are the units of surface tension) should be of order

$$T \sim \frac{pE}{r^2} \sim \frac{1}{4\pi\epsilon_0}\frac{p^2}{r^5} = 0.14 \text{ J m}^{-2} \qquad (141)$$

The measured value of T for water is in fact 0.074 J m^{-2}. Notice that surface tension units may also be expressed as "Newtons per meter," a force per unit length.

The latter interpretation is useful for the current problem. If the surface is displaced by a small amount ξ_z, the vertical component of the tension force per unit length is $T(\partial\xi_z/\partial x)$, where x axis is along the surface, and T is the surface tension. A surface segment of extension δx has forces pulling in opposite senses at x and $x + \delta x$. The net force per unit length is

$$T(\partial\xi_z/\partial x)_{x+\delta x} - T(\partial\xi_z/\partial x)_x \simeq T(\partial^2\xi_z/\partial x^2)(\delta x),$$

analogous to waves on a string. This force is directed downward if the curvature is negative. The opposing pressure (force per unit area) is thus $-T(\partial^2\xi_z/\partial x^2)$, i.e., the pressure is increased at the surface if the tension force is directed downwards.

When the surface of water is distorted, the local Lagrangian pressure change is therefore no longer zero. Instead, it takes on the value dictated by the surface tension:

$$\Delta P = -T\frac{\partial^2\xi_z}{\partial x^2} = \delta P + \xi_z\frac{\partial P}{\partial z} \qquad (142)$$

With the pressure gradient equal to $-\rho g$ and ξ_z proportional to e^{ikx}, we obtain

$$\delta P = \rho\xi_z\left(g + Tk^2/\rho\right). \qquad (143)$$

Since this is the only point in the analysis in which the gravitational field g appears (remember that it disappeared from the linear equations of motion), the entire previous theory is unmodified, except for replacing g by $g_{eff} = g + Tk^2/\rho$! For water, T is conveniently measured in cgs units (74 ergs per cm^2), and the dimensionless parameter that measures the relative importance of surface tension $Tk^2/(\rho g)$ is equal to unity at a wavelength of about 1.7 cm. Our dispersion relation becomes:

$$\omega^2 = g_{eff}k\tanh(kH) = \left(gk + \frac{Tk^3}{\rho}\right)\tanh(kH) \qquad (144)$$

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In the deep water limit, the wave velocity is

$$\frac{\omega}{k} = \left(\frac{g}{k} + \frac{Tk}{\rho} \right)^{1/2}. \tag{145}$$

Long wavelength (surface gravity waves) and short wavelength (capillary waves) disturbances have complementary properties. Surface gravity wave crests move more rapidly at longer wavelengths (because long wavelength disturbances are farther from precise pressure balance), whereas capillary waves move more rapidly at smaller wavelengths (because short wavelength disturbances have a greater restoring tension). Raindrops on a pond excite capillary waves and the circular wave pattern emanating from the contact point shows smaller and smaller separation of crests at larger distances, since the short wavelengths move fastest. A big stone "kerplooshing" in a lake excites surface gravity waves, and the circular pattern shows larger and larger separation of crests at greater distances, because long wavelength gravity waves have moved the farthest.

4.3   Sound Waves in One Dimension

Compressional disturbances in a uniform medium propagate as sound, or acoustic, waves. As noted in the previous section, sound waves are nondispersive: all frequency components travel at the same velocity. (Imagine listening to a symphony orchestra if this were not the case!) The primary point of interest in this relatively simple calculation is to derive an expression for the speed of sound in a gas.

The equations of an adiabatic gas in one dimension are

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0 \tag{146}$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \tag{147}$$

$$P = K \rho^{\gamma} \tag{148}$$

Our initial state will be the simplest possible: P and ρ both constant, $v = 0$. We introduce Eulerian linear perturbations to all flow variables. The linearized equations become

$$\frac{\partial}{\partial t} \frac{\delta \rho}{\rho} + \frac{\partial \delta v}{\partial x} = 0 \tag{149}$$

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$$\frac{\partial \delta v}{\partial t} = -\frac{1}{\rho}\frac{\partial \delta P}{\partial x} \tag{150}$$

$$\frac{\delta P}{P} = \gamma \frac{\delta \rho}{\rho} \tag{151}$$

Replacing δP in favor of $\delta \rho$ in the middle equation above gives

$$\frac{\partial \delta v}{\partial t} = -a^2 \frac{\partial}{\partial x}\frac{\delta \rho}{\rho} \tag{152}$$

where

$$a^2 = \gamma \frac{P}{\rho}. \tag{153}$$

a^2 is in fact the square of the speed of sound, as we shall presently show.

Differentiating (150) with respect to t and (149) with respect to x, then eliminating the mixed partial derivative leads to the classical wave equation

$$\frac{\partial^2 \delta v}{\partial t^2} = a^2 \frac{\partial^2 \delta v}{\partial x^2} \tag{154}$$

The most general solution to this equation is

$$\delta v = C_1 f(x + at) + C_2 g(x - at) \tag{155}$$

where C_1 and C_2 are abitrary constants, and f and g are arbitrary functions. The function f remains unchanged when $dx/dt = -a$ and represents a disturbance traveling toward negative x at the speed a, while g represents the same thing for a disturbance traveling toward positive x at velocity a. Clearly a is the characteristic sound velocity at which all perturbations travel, since it is easy to show that $\delta \rho$ and δP also depend only upon the arguments $x \pm at$.

Sound waves allow internal communication within a gas (or a solid). If a small disturbance occurs at one location, for example, another location at a distance x away cannot be affected by this disturbance until a time x/a has passed. In other words, sound waves "causally connect" spatially separated mechanical processes.

The wave equation can be written in a form which is suggestive of energy conservation. Multiply (154) by $\partial \delta v/\partial t$. Note that

$$\frac{\partial \delta v}{\partial t}\frac{\partial^2 \delta v}{\partial t^2} = \frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial \delta v}{\partial t}\right)^2\right] \tag{156}$$

Only slightly more work is needed to obtain

$$a^2 \frac{\partial \delta v}{\partial t} \frac{\partial^2 \delta v}{\partial x^2} = -\frac{\partial}{\partial t}\left[\frac{a^2}{2}\left(\frac{\partial \delta v}{\partial x}\right)^2\right] + \frac{\partial}{\partial x}\left(a^2 \frac{\partial \delta v}{\partial t} \frac{\partial \delta v}{\partial x}\right) \qquad (157)$$

(Here, we have assumed that a^2 is a constant in the background.) Using these results, equation (154) transforms to

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\frac{\partial \delta v}{\partial t}\right)^2 + \frac{a^2}{2}\left(\frac{\partial \delta v}{\partial x}\right)^2\right] - \frac{\partial}{\partial x}\left(a^2 \frac{\partial \delta v}{\partial t} \frac{\partial \delta v}{\partial x}\right) = 0 \qquad (158)$$

This is in "conservation form": the time derivative of some quantity, which acts like a density, plus the divergence of an associated flux, must vanish. Indeed, if either $\partial \delta v/\partial x$ or $\partial \delta v/\partial t$ vanish at the boundary, then the wave flux

$$-\left(a^2 \frac{\partial \delta v}{\partial t} \frac{\partial \delta v}{\partial x}\right)$$

vanishes, and

$$\int \left[\frac{1}{2}\left(\frac{\partial \delta v}{\partial t}\right)^2 + \frac{a^2}{2}\left(\frac{\partial \delta v}{\partial x}\right)^2\right] dx \qquad (159)$$

is conserved. This quantity is itself not the true energy density (it has the wrong dimensions), but is directly proportional to it. The actual conserved energy density is

$$\frac{\rho}{2}\left(\frac{\partial \xi}{\partial t}\right)^2 + \frac{\rho a^2}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 \qquad (160)$$

where $\xi(x)$ is the displacement of the fluid element located at x before it is perturbed. (This quantity is exactly the same as our previous expression except for a factor of ω^2.) The energy flux is similarly modified:

$$-\left(\rho a^2 \frac{\partial \xi}{\partial t} \frac{\partial \xi}{\partial x}\right) \qquad (161)$$

Exercise. For any type of wave disturbance, we will soon see that the energy flux is given by the average of $\delta P \, \delta v$. This is the rate at which the wave pressure does work per unit area of gas. Show that the equation (161) is equivalent to $\delta P \, \delta v$.

4.4 Harmonic Solutions

As with our treatment of water waves, we seek plane wave solutions of the form $\exp(ikx - i\omega t)$, since we may superpose such harmonics to reproduce any initial condition we choose. The wavenumber k and angular frequency ω are related to the wavelength λ and wave frequency ν by $k = 2\pi/\lambda$, $\omega = 2\pi\nu$. Each fluid variable has the same harmonic x, t dependence, but their amplitudes will of course differ. We may use this approach from the start of the analysis, once the linear equations are found, without having to first reduce the problem to one variable. This is a good approximation even when the background is dependent upon x (or more spatial dimensions), provided that our wavenumber is much larger than background spatial gradients. If we wish to work to higher order accuracy however, for example by letting the amplitudes and wavenumbers depend weakly on position, *we must first reduce the problem to a single wave equation.* This is because the precise functional form of the amplitudes of the pertubed density, pressure, and velocity will general differ from one another in a manner that cannot be determined *a priori.* These considerations may be formalized in a mathematical procedure known as WKB theory (for Wentzel–Kramers–Brillouin)[1].

In three-dimensions the solutions have a spatial dependence $\exp(i\boldsymbol{k} \cdot \boldsymbol{r})$ where $\boldsymbol{k}$ and $\boldsymbol{r}$ are vectors. These are called plane waves because the planes of constant $\boldsymbol{k} \cdot \boldsymbol{r}$ all have the same phase. For sound waves, if we take our wave equation (154), and look for harmonic solutions, we find that such solutions exist, provided that

$$\omega^2 = k^2 a^2, \quad k^2 = k_x^2 + k_y^2 + k_z^2 \tag{162}$$

which is our simple dispersion relation.

What is value of a^2? We have

$$a^2 = \gamma\frac{P}{\rho} = \gamma\frac{nkT}{mn} = \gamma\frac{kT}{m} \tag{163}$$

where n is the number density of particles, k is the Boltzmann constant, T is the temperature, and m is the average mass per particle. For a monatomic gas $\gamma = 5/3$, for a diatomic molecule like H_2, $\gamma = 7/5$. The essential point is that the speed of sound is roughly the thermal speed of a typical gas atom or molecule. In a mixture of different gases, say a dry nitrogen-oxygen atmosphere, n is the total number density of particles of all species, and

$$a^2 = \gamma\frac{nkT}{m_N n_N + m_O n_O} \tag{164}$$

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[1]The theory was actually developed by Liouville, Rayleigh, and Jeffreys, so it should be called LRJ theory! The technique became well-known after the advent of quantum mechanics when it was introduced by Wentzel, Kramers, Brillouin, and others.

where the subscript N refers to Nitrogen and O to Oxygen. With $T = 300$K, $n_N/n = 0.79$, $n_O/n = 0.21$, and molecular masses taken from a table of atomic weights (nitrogen, in particular, is a mixture of different isotopes), one finds $a^2 = 348$ meters per second. Atmospheric sound waves therefore take about 3 seconds to travel one kilometer. (Under realistic conditions, water vapor in the air can also measurably affect the speed of sound. This effect is hard to calculate, because atmospheric water vapor is not purely molecular; it is present on scales from individual molecules to macroscopic [but still tiny] droplets.)

Exercise. Show that for a harmonic acoustic wave, the energy flux is the energy density multiplied by a. What does this imply for the speed at which wave energy is propagated in a sound wave?

4.5 Incompressible Waves: the Boussinesq Approximation

For sound waves, the restoring force is provided by the outward directed pressure of compression, or the inward directed pressure of the ambient gas on the rarified gas during its dilation. In either case, it is the internal pressure force that tries to restore equilibrium. When the restoring force is purely external—gravity, for example—the flow generally behaves as though the fluid were nearly incompressible. For surface water waves (gravitational restoring force), we treated the flow as incompressible, but in this case the physical fluid itself was truly incompressible! Gravity can be the restoring force in gaseous systems as well, and the resulting waves set up a nearly incompressible velocity flow, even though the gas itself is compressible. This class of waves, which includes internal buoyant atmospheric modes (and corresponding oceanic modes), is of great meteorolgical importance. The question we need to address is what does "nearly incompressible" mean for a gas, in a quantitative sense? The compressibility is small compared to *what*?

Compressibility arises principally from the density changes associated with a diverging velocity flow,

$$-i\omega\frac{\delta\rho}{\rho} = -\boldsymbol{\nabla}\cdot\delta\boldsymbol{v} = -i\boldsymbol{k}\cdot\boldsymbol{\delta v}$$

In the so-called **Boussinesq** approximation, we ignore these density changes, and the flow behaves as though it were incompressible, with a vanishing velocity divergence. More precisely, if $k\delta v$ is the magnitude of $\boldsymbol{k}$ times the magnitude $\boldsymbol{\delta v}$, we are really assuming that

$$\frac{\delta\rho}{\rho} \ll \left|\frac{k\delta v}{\omega}\right| \quad \text{(Boussinesq Approximation)}$$

Physically, a sufficiently slowly-moving fluid element remains in close pressure equilibrium with its surroundings. If the density of the fluid must change to meet this requirement, then it will do so. A density change does not, by itself, contradict the above Boussinesq inequality: at large k the right side of the equation can become arbitrarily large, since ω is typically bounded. If L is the characteristic length scale of a background gradient, the frequencies of interest will ordinarily be of order a/L for a Boussinesq wave, as opposed to ka for a compressive wave. To be precise, we shall be working in a regime in which the following hierarchy of perturbation scales holds:

$$\frac{\delta P}{P} \sim \frac{1}{kL}\frac{\delta \rho}{\rho} \ll \frac{\delta \rho}{\rho} \sim \frac{\delta v}{a} \ll \frac{k\delta v}{\omega} \sim (kL)\frac{\delta v}{a} \tag{165}$$

To see how this works in detail, we derive the dispersion relation of incompressible wave modes in a gravitationally stratified medium with an entropy gradient. For a liquid, we may take the limit in which the adiabatic index approaches infinity, in which case the waves are caused by density gradients. (The density in an ocean may depend upon height, for example, because of varying salinity. Salt water is heavier than fresh water.) In both cases, the disturbances are called *internal waves.* They are sometimes referred to as *internal gravity waves*, especially in water, to distinguish them from surface gravity waves. They are important not only for understanding the terrestrial atmosphere and oceans, but the interiors of stars as well. (Note that in astrophysical texts, "gravity waves" can sometimes mean gravitational radiation. The context is usually clear.)

4.5.1 Internal Waves

Consider a medium in which the gravitational field points in the $-z$ direction. There are entropy, pressure, and density gradients with respect to z. We introduce a z velocity perturbation of the form

$$\delta v_z \exp(ikx - i\omega t), \tag{166}$$

where x is in a direction perpendicular to z, but otherwise arbitrary. The equilibrium background is motionless. The linearized equations of motion are

$$-i\omega \delta v_z = \frac{\delta \rho}{\rho^2}\frac{dP}{dz} \tag{167}$$

$$i\gamma \omega \frac{\delta \rho}{\rho} + \delta v_z \frac{d\ln P\rho^{-\gamma}}{dz} = 0 \tag{168}$$

(Since $\delta v_x = 0$, we find $\delta P = 0$ from the x equation of motion.) Note that both mass conservation and the Boussinesq approximation are automatically

satisified ($\boldsymbol{\nabla}\!\cdot\!\delta\boldsymbol{v} = 0$). The second equation above is simply the perturbed adiabatic form of the entropy equation,

$$\delta\left(\frac{d\ln P\rho^{-\gamma}}{dt}\right) = 0 \tag{169}$$

coupled with $\delta P = 0$. This is such a simple system that the k dependence has disappeared completely, and the resulting dispersion relation of equations (167) and (168) is:

$$\omega^2 \equiv N^2 = -\frac{1}{\gamma\rho}\frac{dP}{dz}\frac{d\ln P\rho^{-\gamma}}{dz} \tag{170}$$

This defines the so-called Brunt-Väisälä frequency N. Vertical displacements in a vertically stratified medium oscillate at a frequency N which is independent of their horizontal wavenumber k. Physically, these oscillations come about because an upward adiabatic displacement of an element of fluid leaves the element cool relative to its surroundings—if, and this is an important "if", the background entropy is increasing upward. (The pressure, on the other hand, must *always* decrease upward to oppose the force of gravity.) A cool element is denser than its surroudings (because of local pressure balance), so the buoyant force is restoring. If, on the other hand, the background entropy gradient were to decrease upwards, we would find that $N^2 < 0$! This precludes a wave respose altogether, and results instead in a fluid *instability*. An upward adiabatic displacement now produces a fluid element that is warmer than its surroundings (the surroundings have cooled more rapidly than adiabatic as one moves up), so there is no restoring buoyant behavior. Instead, the displaced element continues upward, where the anti-buoyancy is even worse. The resulting instability gives rise to thermal convection, the upward transport of heat by a turbulent fluid. You've seen this in the kitchen: it is boiling water. The sun "boils" as well in its outer layers, producing a convection zone which is clearly visible as regions of ascending hot gas and decending cool gas on the solar surface. We shall have more to say on convection when we discuss instabilities more formally later in the course.

Our Brunt-Väisälä dispersion relation does not depend upon wavenumber, which is curious. It means, in fact, that there is no real wave propagation: as we shall shortly see, $d\omega/dk$ cannot vanish if a signal truly propagates. Recall that in our calculation, $\delta P = 0$. With exact pressure equilibrium being maintained everywhere, it is not surprising that nothing is propagating. We need to find a way to get real waves, not just stationary oscillations.

Consider more general disturbances of the form

$$\boldsymbol{\delta v}\exp[i(k_z z + k_x x) - i\omega t] \tag{171}$$

The linearized equations are then

$$k_x\delta v_x + k_z\delta v_z = 0 \quad \text{(mass conservation)} \tag{172}$$

$$-i\omega\,\delta v_x = -ik_x\frac{\delta P}{\rho} \quad \text{(x-eqn. of motion)} \tag{173}$$

$$-i\omega\,\delta v_z = -ik_z\frac{\delta P}{\rho} + \frac{\delta\rho}{\rho^2}\frac{dP}{dz} \quad \text{(z-eqn. of motion)} \tag{174}$$

$$i\gamma\omega\frac{\delta\rho}{\rho} + \delta v_z\frac{d\ln P\rho^{-\gamma}}{dz} = 0 \quad \text{(entropy eqn.)} \tag{175}$$

It is a straightforward matter to extract the dispersion relation

$$\omega^2 = \frac{k_x^2}{k^2}N^2 \tag{176}$$

where $k^2 = k_x^2 + k_z^2$. These are true internal waves, exhibiting not only dispersion (different velocities for different wavenumbers), but non-isotropic behavior as well (ω depends upon the *direction* of $\boldsymbol{k}$).

We have noted that internal waves are of importance to geophysicists, because they propagate through the atmosphere and in the interior of oceans. In the ocean, the entropy equation is replaced by condition $d\rho/dt = 0$. Alternatively, the resulting N^2 may also be obtained by taking the incompressible limit $\gamma \to \infty$ in our adiabatic formula:

$$N^2 = -\frac{1}{\gamma\rho}\frac{dP}{dz}\frac{d\ln P\rho^{-\gamma}}{dz} \stackrel{\gamma\to\infty}{=} \frac{1}{\rho}\frac{dP}{dz}\frac{d\ln\rho}{dz} \quad \text{(Incompressible Fluid)} \tag{177}$$

Note that it is the density of each fluid element that remains fixed in an incompressible fluid, as opposed to the entropy in a gas. Do not confuse $d\rho/dt = 0$ with $\rho =$ constant. Spatial gradients of ρ are necessary for internal waves to exist. It is the pre-existing background density gradient (due to changing salinity) that allows these oceanic buoyant waves to propagate.

In stars similar to the sun, the convectively unstable zone is in the star's outer layers. Turbulence maintains a value of N^2 very close to zero, but slightly negative to maintain a low level of instability. Internal waves must propagate below the convective zone, deeper in the solar interior. The amplitudes caused by such waves at the star's surface are therefore very small. By way of contrast, sound waves propagate freely in the convective zone, and those that are reflecting from just below the sun's surface are readily detected. They have been used as diagnostic probes of conditions in the solar interior, while internal waves remain invisible. This is a pity, because such waves could convey information about the very deep interior of the sun, in regions inaccessible to sound wave probes. Internal waves are also excited in stars by tidal interactions in binary systems, where they are thought to play a central role in the determining how the stellar orbits evolve.

Exercise. Verify that the scalings of equation (165) are valid for internal waves.

4.5.2 Rossby Waves

Our next example is an interesting class of waves that propagate in planetary atmospheres, and owe their existence to vorticity conservation. They have the remarkable property of propagating in only one direction! These are known as "Rossby waves," after the meteorologist Carl Rossby who first studied the phenomenon in the late 1930's.

We confine ourselves to a small local plane on the earth's surface, rotating with a 24 hour period. To obtain the equations of motion under these circumstances is very simple, and is just a matter of adding in the Coriolis term. (Centrifugal forces are small in this problem.) The equation of motion is

$$\frac{D\boldsymbol{v}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P \tag{178}$$

We assume that an enthalpy $\mathcal{H}$ exists, $\rho d\mathcal{H} = dP$, and that the right hand side is then the excess enthalpy caused by the wave's presence. Only motions along the surface of the planet enter our problem. In spherical coordinates, we thus consider the θ and ϕ equations of motion. The angular velocity $\boldsymbol{\Omega}$ is along the axis of rotation, but only the radial component $\Omega\cos\theta$ perpendicular to the planet's surface enters into the equations. Let us call the local θ direction $dx = r d\theta$, and the local ϕ direction $dy = r\sin\theta \, d\phi$. Because we are considering a small patch of the earth's surface, the earth's radius r is regarded as very large, $1/r$ is very small, and the curvature terms in our equations may be ignored. Thus, x and y are treated as Cartesian variables. Our linearized equations are

$$\frac{\partial \delta v_x}{\partial t} - f\delta v_y = -\frac{\partial \mathcal{H}}{\partial x} \tag{179}$$

$$\frac{\partial \delta v_y}{\partial t} + f\delta v_x = -\frac{\partial \mathcal{H}}{\partial y} \tag{180}$$

where $f = 2\Omega\cos\theta$ is the so-called Coriolis parameter. Differentiating the top equation by y and the bottom by x and equating the mixed enthalpy derivatives gives

$$\frac{\partial}{\partial t}\left(\frac{\partial \delta v_x}{\partial y}\right) - f\frac{\partial \delta v_y}{\partial y} = \frac{\partial}{\partial t}\left(\frac{\partial \delta v_y}{\partial x}\right) + f\frac{\partial \delta v_x}{\partial x} + \delta v_x\frac{df}{dx} \tag{181}$$

Notice that the term $df/dx = (1/r)df/d\theta = -2\Omega\sin\theta/r$, is crucial to this problem. Rossby waves depend upon a varying Coriolis parameter. The mass conservation equation

$$\boldsymbol{\nabla}\cdot\boldsymbol{\delta v} = 0 \tag{182}$$

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can be most easily accommodated by the use of a "stream function" ψ:

$$\delta v_x = \frac{\partial \psi}{\partial y}, \qquad \delta v_y = -\frac{\partial \psi}{\partial x} \tag{183}$$

Then equation (181) simplifies nicely:

$$\frac{\partial}{\partial t} \nabla^2 \psi = \frac{\partial \psi}{\partial y} \frac{df}{dx} \tag{184}$$

Now we may try a plane wave solution of the form $\exp i(k_x x + k_y y - \omega t)$. Notice the rather subtle point that we must *not* regard f as a constant in this problem, so that some care is needed to verify that this plane wave satisfies the original equations (179) and (180). You should try to do this. How does $\mathcal{H}$ behave?

The dispersion relation is

$$\omega = \frac{k_y (df/dx)}{k_x^2 + k_y^2} = -\frac{2\Omega k_y \sin\theta}{k^2 r}, \tag{185}$$

where $k^2 = k_x^2 + k_y^2$. Notice that it is linear in both in Ω and k_y. With $\Omega > 0$, $\sin\theta > 0$, we must have $k_y < 0$. In a Rossby wave, surfaces of constant phase must drift in the direction of decreasing ϕ, i.e., westward. Rossby waves are "sens unique!"

The physical origin of a Rossby waves is vorticity conservation. Potential vorticity is conserved in two-dimensions. In a rotating frame, it is the sum of the intrinsic vorticity of a fluid element plus the local vorticity of the rotating earth that is conserved:

$$\frac{D}{Dt} (\varpi + 2\Omega \cos\theta) = 0 \tag{186}$$

where

$$\varpi = \frac{\partial \delta v_y}{\partial x} - \frac{\partial \delta v_x}{\partial y}, \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{\delta v} \cdot \boldsymbol{\nabla} \tag{187}$$

(*Exercise:* Prove this.) Thus, in the northern hemisphere, when an element moves toward the equator, $2\Omega \cos\theta$ decreases, and the intrinsic vorticity ϖ goes up; when the element moves northward, ϖ goes down. It is these compensating, Coriolis-driven motions that produce the westward drift of the wave. The displacement of a fluid element thus follows the phase of the vorticity: a northward (negative δx) displacement is associated with a negative vorticity. In figure 3, we have sketched one full wavelength of a northern hemisphere Rossby wave. The left half-wavelength corresponds
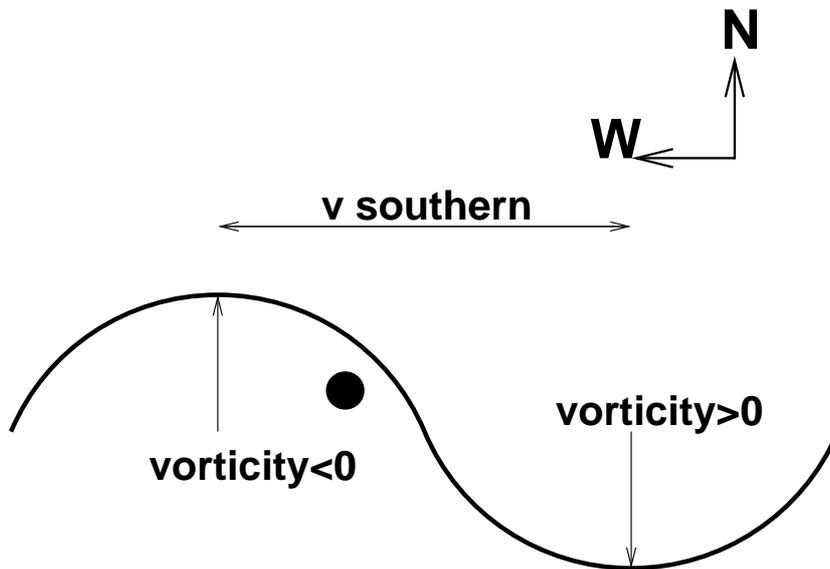
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Figure 3: Why Rossby wave crests drift westward. The figure shows a small portion of the northern hemisphere. The S-curve inidcates an organized wave of fluid displacements. The black dot is in a region of northward displacements and negative (intrinsic) vorticity, but the dot's velocity is southern — a northern velocity would be inconsistent with the sign of the vorticity (do you see why?) — so the dot's vorticity must be increasing with time. This means that the adjacent region to the east will soon be arriving: the wave pattern drifts westward.

to northern displacements, and is a region of negative vorticity; the right half-wavelength corresponds to southern displacements and positive vorticity. South-directed velocity, which remember is positive in these coordinates, is present in both positive and negative vorticity regions, as shown in the figure. The black dot is in a region where the displacement is northward, but the velocity is southward, and it will therefore soon find its vorticity increasing. The eastward adjacent region of positive vorticity will therefore move westward, bringing its increased vorticity to the black dot. The whole wave therefore drifts westward.

In the 1930's, after the discovery of these waves, there was great hope that weather forecasting would be vastly improved. In fact, we now know that such grand hopes were naïve (weather is chaotic), but large scale climactic influences are, on large enough time scales, closely related to Rossby wave circulations in both the atmosphere and ocean. The flow of both the jet

stream and Gulf stream are influenced by slowly moving Rossby waves, as is the El Niño phenomenon. The latter, in particular, can significantly affect the climate on time scales of years.

4.5.3 Long Waves

We end by coming back to surface water waves, this time with periods sufficiently long to be influenced by the rotation of the earth. These differ from Rossby modes in having significant vertical motions. The restoring force is a combination of gravity and Coriolis motions. If we write down the fundamental equation for mass conservation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \tag{188}$$

we may note that there is a peculiar feature: even though the density is formally constant, $\partial \rho / \partial t$ becomes infinite when the water surface rises above a level at which it was previously absent! We may avoid this purely formal singularity by integrating vertically over the height of the water. If the undisturbed level is h and the total level is $h + \zeta$ ($\zeta \ll h$), then the linearized mass conservation equation is

$$\frac{\partial \rho \zeta}{\partial t} + \frac{\partial (\rho h v_x)}{\partial x} + \frac{\partial (\rho h v_y)}{\partial y} = 0 \tag{189}$$

since $\boldsymbol{v}$ is already a small quantity. (Note that we work in the asymptotic long wave limit $kh \ll 1$, so that the velocity $\boldsymbol{v}$ is independent of depth.) This height-integrated equation is now quite regular, and the density may be scaled out:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h v_x)}{\partial x} + \frac{\partial (h v_y)}{\partial y} = 0 \tag{190}$$

An incompressible three-dimensional fluid looks a compressible fluid in two dimensions: squeezing the fluid in two dimensions makes it rise into the third, thereby increasing the local *surface* density.

The excess pressure is simply the weight per unit area of the height of the water above (or below) h, i.e., $\rho g \zeta$. Our two Coriolis equations are

$$\frac{\partial v_x}{\partial t} - f v_y = -\frac{\partial g \zeta}{\partial x} \tag{191}$$

$$\frac{\partial v_y}{\partial t} + f v_x = -\frac{\partial g \zeta}{\partial y} \tag{192}$$

Unlike the case of Rossby waves, the leading order behavior of long waves allows us to treat f as a constant. Assuming that all variables have the plane wave form $\exp i(k_x x + k_y y - \omega t)$ produces the dispersion relation

$$\omega^2 = k^2 gh + f^2 \tag{193}$$

where $k^2 = k_x^2 + k_y^2$. Thus, f is a lower limit "cut-off" frequency of any long wave. This is not a problem for tsunamis, whose frequency is well above f, but matters are more interesting for waves excited by the tidal forces of the moon. The earth turns 2π radians per day, and the moon orbits around the earth in the same sense $2\pi/29$ radians per day. Thus, the frequency of the tidal forcing is

$$\omega_t = 2 \times \left(2\pi - \frac{2\pi}{29} \right).$$

The f parameter is $4\pi \sin \lambda$, where λ is the latitude angle measured from the equator. f is less than ω_t everywhere except for $\lambda > 75$ degrees, at polar latitudes. Everywhere else on the earth's surface, tidal long waves may be excited. Understanding the propagation properties of tidal long waves is particularly important when these ocean waves flow from deeper oceans into shallow coastal seas and large amplitudes may result.

Exercise. The variation in the Coriolis parameter, which was critical to the existence of Rossby waves, also allows a type of trapped long wave to form, known as a *Kelvin wave.* Unlike Rossby waves, which travel only westward, Kelvin waves travel only eastward!

1. Consider flow near the equator, and assume that $\delta v_x = 0$, so that only flow parallel to latitudes occurs ("zonal flow"). Let $\alpha = (\pi/2) - \theta$, the latitude angle. Show that the equations of motion (191) and (192) may be written

$$-2\Omega \, \delta v_y \alpha = \frac{g}{r} \frac{\partial \zeta}{\partial \alpha}$$

$$\frac{\partial \delta v_y}{\partial t} = -g \frac{\partial \zeta}{\partial y}$$

where r is the earth's radius, and that mass conservation becomes

$$\frac{\partial \zeta}{\partial t} + h \frac{\partial \delta v_y}{\partial y} = 0$$

2. Solve these equations and show that a plane wave of the form $\exp(iky - i\omega t)$ has the dispersion relation

$$\omega^2 = ghk^2,$$

and an amplitude dependence of

$$\exp\left(-\frac{kr\Omega\alpha^2}{\omega}\right)$$

Why does this require only eastward propagating waves?

Kelvin and Rossby waves are intimately involved with the El Niño phenomenon, in which unusally warm ocean temperatures off the Pacific coast of South American cause worldwide climate changes. Under normal conditions, there is a strong east-to-west moving wind between, say, Chile and Tahiti, due to the effects of the Coriolis force on atmospheric convective circulation. This wind causes ocean surface currents to flow westward from Chile, drawing up cold subsurface layers of water. The coastal waters are thereby cooled. If this wind should weaken for some reason, this oceanic upwelling would also weaken, and the surface temperatures would rise relative to the cool water that normally is present. But this change in surface temperature generates westward moving Rossby waves, which in turn propagate the warmer surfaces temperatures across the Pacific. This has the consequence of easing the atmospheric east-to-west pressure gradient force, further weaking the surface wind. A small change in the Chilean surface water conditions is thus amplified and re-amplified. This is the El Niño phenomenon.

How does this stop? The El Niño generated Rossby waves are reflected off the coast in southeast Asia, and return as eastward moving Kelvin waves! The Kelvin waves restore high pressure conditions off the Chilean coast, the east-to-west winds return, and normal conditions are restored. Because the Rossby waves move so slowly, the process can take 3-4 years to complete. See "The physics of El Niño,"

http : //physicsweb.org/articles/world/11/8/8.

4.6   Group Velocity

The dispersion relation (193) has the interesting property of reducing to $\omega = f$ as k (or h) vanishes. f just depends on the rotation of the earth. Can a rotating earth by itself propagate waves, even with no water??

Obviously not. Yet this is truly the velocity at which a $k = 0$ wave crest would propate, ω/k. The point is that this is the velocity at which a point of constant *phase* propagates, and it is only the speed of a "pattern". Another system which has the same form of dispersion relation is a line of pendula whose masses are connected by springs. If the pendula have length l, the dispersion relation is

$$\omega^2 = k^2 C^2 + g/l, \tag{194}$$

where C^2 is the spring tension divided by the mass per unit length and g is the gravitational field. If we cut the springs, the pendula all oscillate, but clearly nothing physical is propagating, even though a pattern will appear to move if we set up the pendula properly. Only when the springs are present will a real signal—in fact, real energy—be propagated. Note that when the springs are present (or when the water has a finite depth!), the frequency depends upon k. That is the key point.

All physical quantities in a wave, including causality itself, propagate not at the phase velocity ω/k, but at what is known as the *group velocity* $\partial\omega/\partial k$. The wave frequency must depend upon the wavenumber if anything physical is being propagated! If ω is independent of k, the group velocity vanishes, and there is no true physical propagation. To understand where the name "group" comes from, imagine a wave packet whose mathematical form is taken to be

$$\exp[iS(x,t)]. \tag{195}$$

At a time t and location x, a local wavenumber and frequency may be defined:

$$k = \frac{\partial S}{\partial x}, \qquad \omega = -\frac{\partial S}{\partial t}. \tag{196}$$

(More generally, $\boldsymbol{k} = \boldsymbol{\nabla} S$, implying that the local wavevector is orthogonal to constant phase surfaces.) Hence

$$\frac{\partial k}{\partial t} = -\frac{\partial \omega}{\partial x} = -\frac{\partial \omega}{\partial k}\frac{\partial k}{\partial x} \tag{197}$$

This, in turn, implies,

$$\frac{\partial k}{\partial t} + v_g \frac{\partial k}{\partial x} = 0, \tag{198}$$

where

$$v_g = \partial\omega/\partial k \tag{199}$$

is the group velocity. Equation (198) states that a "group" of wave crests with well-defined wavenumber k moves along coherently at the velocity v_g. An initial superposition of different wavenumbers would spread apart, each wavenumber moving with its own value of v_g—unless, of course, v_g is a constant, as it is for both sound and light waves.

Exercise. Show that in three dimensions

$$\frac{\partial \boldsymbol{k}}{\partial t} + (\boldsymbol{v_g}\boldsymbol{\cdot}\boldsymbol{\nabla})\boldsymbol{k} = 0$$

where $\boldsymbol{k} = \boldsymbol{\nabla} S$ and $\boldsymbol{v_g} = \boldsymbol{\nabla}_{\boldsymbol{k}}\omega$.

Another simple approach is to consider the superposition of two plane waves with nearly equal wavenumbers k_1 and k_2, and frequencies ω_1 and ω_2. Let $S_1(x,t) = k_1 x - \omega_1 t$, and similarly for 2. Then

$$\exp(iS_1) + \exp(iS_2) = \exp\frac{i(S_2+S_1)}{2}\left[\exp\frac{i(S_1-S_2)}{2} + \exp\frac{i(S_2-S_1)}{2}\right]$$
(200)

The right side is just

$$= 2\left[\cos\left(\frac{S_1-S_2}{2}\right)\right]\exp\frac{i(S_2+S_1)}{2} = 2\cos\left(\frac{S_1-S_2}{2}\right)\exp(iS_{avg}) \quad (201)$$

where S_{avg} is the average of S_2 and S_1. This is a plane wave modulated by a slowly-varying cosine envelope. The modulation has a phase which remains constant along the trajectory

$$x - \frac{\omega_1-\omega_2}{k_1-k_2}\times t = x - v_g t = \text{constant} \tag{202}$$

The modulation therefore travels along the rapidly oscillating plane wave at the group velocity v_g. Since a pure, unmodulated plane wave sends no signal (it has no beginning and no end), true signals travel at the group velocity.

Finally, we illustrate this point by a more rigorous procedure. To understand the mathematics, consider an integral of the form

$$\int_{-\infty}^{\infty} \exp(iM\Phi(x))\,dx \tag{203}$$

as $M \to \infty$. Let Φ be well-behaved with an extremum (it could be a maximum or minimum) at $x = x_m$. Away from $x = x_m$, there is essentially no contribution to the integral, because the oscillations become infinitely rapid as M increases without bound. In the neighborhood of $x = x_m$, however, the oscillations cease, because Φ has no first derivative. The phase is said to be stationary. Thus, the integral may be well-approximated by restricting its range to a small neighborhood around x_m, and expanding Φ in a Taylor series:

$$\int_{x_m-\epsilon}^{x_m+\epsilon} \exp(iM\Phi(x))\,dx = \int_{x_m-\epsilon}^{x_m+\epsilon} \exp[iM[\Phi(x_m) + \Phi''(x_m)(x-x_m)^2/2]\,dx$$
(204)

where $\Phi''(x_m)$ is the nonvanishing second derivative of Φ at $x = x_m$. But in fact, there is little error in taking the range of integration back to $\pm\infty$, since there is still, once again, little contribution to the integral away from $x = x_m$. We are left with

$$\exp[iM\Phi(x_m)] \int_{-\infty}^{\infty} e^{i\Phi''(x_m)s^2/2}\,ds \tag{205}$$

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where we have shifted the integration variable to $s = x - x_m$. The remaining integral may be found in tables or evaluated by straightforward contour integration techniques. It is equal to $e^{\pm i\pi/4}(\pi/|\Phi''(x_m)|)^{1/2}$ where the $\pm$ sign is the sign of $\Phi''(x_m)$. This technique of evaluating an integral in which the argument is rapidly varying is called the method of stationary phase.

For our application, we consider a Fourier superposition of waves of different wavenumbers:

$$\int_{-\infty}^{\infty} A(k) \exp[i(kx - \omega t)] \, dk \tag{206}$$

It is assumed that the Fourier transform $A(k)$ is not a rapidly varying function of k. At sufficiently large values of the exponential argument (either x, t or k, ω), we may use the method of stationary phase to evaluate this integral. We need not find its exact value. Rather, we note that its value will be completely dominated by the contribution at the point at which the exponential argument has a maximum (or minimum) with respect to k:

$$x - \frac{d\omega}{dk} t = 0 \tag{207}$$

In other words, the space curve $x = v_g t$ will follow the region where all of the contribution to the Fourier integral originates. The group velocity tracks the wave packet.

The above considerations show that the physical attributes propagate at the group velocity $d\omega/dk$. In two or three dimensions, this generalizes to the gradient in wavenumber space of ω. (The method of stationary phase requires that all three directions correspond to an extremum of the argument of the exponential function.) In other words, the group velocity always points in the direction orthogonal to surfaces of constant ω.

As a slightly esoteric but very physical example of this, consider quantum mechanical de Broglie waves. For particles in a potential $V(\boldsymbol{r})$,

$$E = \hbar\omega = \frac{\hbar^2 k^2}{2m} + V \tag{208}$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$. Thus

$$\boldsymbol{v_{gr}} = \boldsymbol{\nabla}_{\boldsymbol{k}} \omega = \frac{\hbar \boldsymbol{k}}{m} = \frac{\boldsymbol{p}}{m} \tag{209}$$

where $\boldsymbol{p}$ is the momentum. This corresponds to the classical velocity of a particle, which indeed does carry its physical attributes! Note that the phase velocity contains nothing of physical significance here.

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Our intuitive understanding of group velocity has been based on the notion of a wave packet with different frequencies and wavenumbers clustered around a dominant frequency ω. But in practice, we work with a signal propagation velocity v_g even when only a pure harmonic wave $\exp(ikx - i\omega t)$ is present in our problem. If there are no other neighboring frequencies present, how do we make sense of the concept of a group velocity?

The point, which is subtle, is that there are always neighboring frequencies present, even when there do not seem to be. Any signal must have been turned on a some point in time, as we have already emphasized. A "pure" harmonic wave had to have been "turned on" at some time in the past, however distant that time may have been ($t \to -\infty$). Accordingly, it is of interest to consider the behavior of a wave with frequency $\omega + i\gamma$, and then *let $\gamma > 0$ approach zero at the end of the calculation.* The fact that we approach ω from the positive imaginary direction is not lost in this limit; it leaves a trace by introducing the group velocity into the structure—the causal structure—of the wave. It leaves this imprint in the form of a group velocity.

Start with a plane wave $\exp(ikx - i\omega t)$, a pure mode propagating in the positive x direction. We introduce a small positive imaginary part $i\gamma$ to ω to account for the wave being turned on from zero amplitude in the distant past. But ω and k are not independent variables, they are related by $k = k(\omega)$, the inverse of the dispersion relation $\omega = \omega(k)$. Hence, if ω changes by $i\gamma$ then k must change by

$$k \to k + i\gamma k' = k + i\gamma/v_g \tag{210}$$

where $k' = dk/d\omega = 1/v_g$. The appearance of the group velocity v_g is intimately linked to the coupling between ω and k that is present by the existence of the dispersion relation.

The wave form becomes

$$\exp[\gamma(t - x/v_g)] \ \exp[(ikx - i\omega t)] \tag{211}$$

Notice that the amplitude of the exponential modulation of the oscillation does not simply rise everywhere uniformly with time. Instead, it *propagates* at the group velocity v_g. At remote times in the past the wave was tiny, and this small amplitude region has now advanced toward large positive x (assuming $k' > 0$), where the wave envelope is indeed vanishing. Turning on the wave is a causal process, and the amplitude could rise uniformly only if there was no change in the wavenumber k accompanying a change in ω; that is, only if ω and k were unrelated by a dispersion relation. The interdependence of k and ω is ultimately responsible for the causal structure of the modulating envelope. The propagation speed v_g is independent of the

vanishing parameter γ. At the end of the next section, you will be asked to show that in the limit $\gamma \to 0$, the wave energy propagates at v_g.

Dispersion relations of the form

$$\omega^2 = k^2 c^2 + \omega_0^2, \tag{212}$$

where c is a characteristic velocity and ω_0 a constant frequency, appear often in physics. We have already seen two examples: long waves in the oceans and seas, and a system of pendula coupled with springs. Other examples include (1) electromagnetic waves in a plasma

$$\omega^2 = k^2 c^2 + \omega_p^2 \tag{213}$$

where now c is the speed of light and ω_p is the natural oscillation frequency (due to separation of charges) in a plasma; and (2) relativistic Klein-Gordon waves,

$$(\hbar\omega)^2 = (\hbar k)^2 c^2 + m^2 c^4 \tag{214}$$

where c once again is the speed of light. In each of these cases, the phase velocity is in excess of c, which in these two examples means faster than the speed of light! The group velocity, on the other hand, is always less than c. For one-dimensional waves in a plasma, for example,

$$v_g = \frac{kc^2}{\omega} = \frac{kc^2}{\sqrt{k^2 c^2 + \omega_p^2}} < c \tag{215}$$

This also guarantees that material particles obeying the Klein-Gordon dispersion relation move slower than the speed of light. This makes physicists happy.

Water waves also show interesting behavior due to group velocity effects. In deep water, $\omega^2 = gk$, and the group velocity of wavenumber k is half as fast as the speed of wave crests at the same k. If you look carefully at waves generated by winds or a passing boat, you will see a spreading packet of waves moving along, and within the packet wave crests will appear out of "nowhere" in the back, move forwards, and then disappear at the front of the packet! As a nice example of the interplay between phase and group velocities, consider the wake behind a boat (or even a duck) moving through the water.

Start with figure 4. From the point of view of the boat, the water is passing by with a velocity V. Waves are generated at the bow of the boat, at all wavelengths for which the boat's velocity exceeds the corresponding phase velocity v_p. In particular, forward propagating waves for which $v_p = V$ would appear stationary just in front of the boat. More generally, at an angle θ
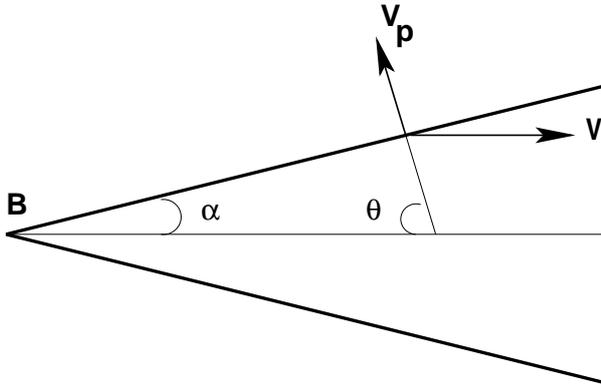
Figure 4: **B** represents a boat with a current of velocity V streaming past. Along a wedge of half opening angle α, wavecrests generated by the boat are stationary if their phase velocity satisfies $V\cos\theta = v_p$, where $\theta + \alpha = 90°$. Different wavenumbers k will have different angles α.

measured relative to the path of the boat (fig. 4), the phase velocity v_p satisfying the equation

$$v_p - V\sin\alpha = 0 \rightarrow V\cos\theta = v_p = (g/k)^{1/2} \qquad (216)$$

picks out the direction for which the component of **V** normal to the wave fronts exactly cancels the outward velocity of the fronts, namely v_p. These waves appear stationary in the frame of the boat. If in time t, the boat traveled Vt and the waves traveled $v_p t = Vt\cos\theta$, then the end of each ray traced by each wave of a given wavenumber k would define a set of points forming a circle with diameter Vt. The points P and Q shown in figure 5 are representative. BUT, waves of a given wavenumber are found *not* at P and Q, but at P' and Q': this is the *group velocity* distance

$$v_g t = v_p t/2 = (Vt/2)\cos\theta$$

from the original position of the boat B'. The actual waves therefore travel in the same direction (e.g., toward P and Q) as the move out from B', but only reach half of the formal phase velocity distance. In other words, the waves form a circle not of diameter Vt, but of diameter $Vt/2$.

The wake of the boat lies within the wedge whose boundary starts from the ship's current position B, and is tangent to this smaller circle. As shown in the second drawing of figure 5, the wedge half angle α satisifes

$$\sin\alpha = \frac{1}{3} \qquad (217)$$

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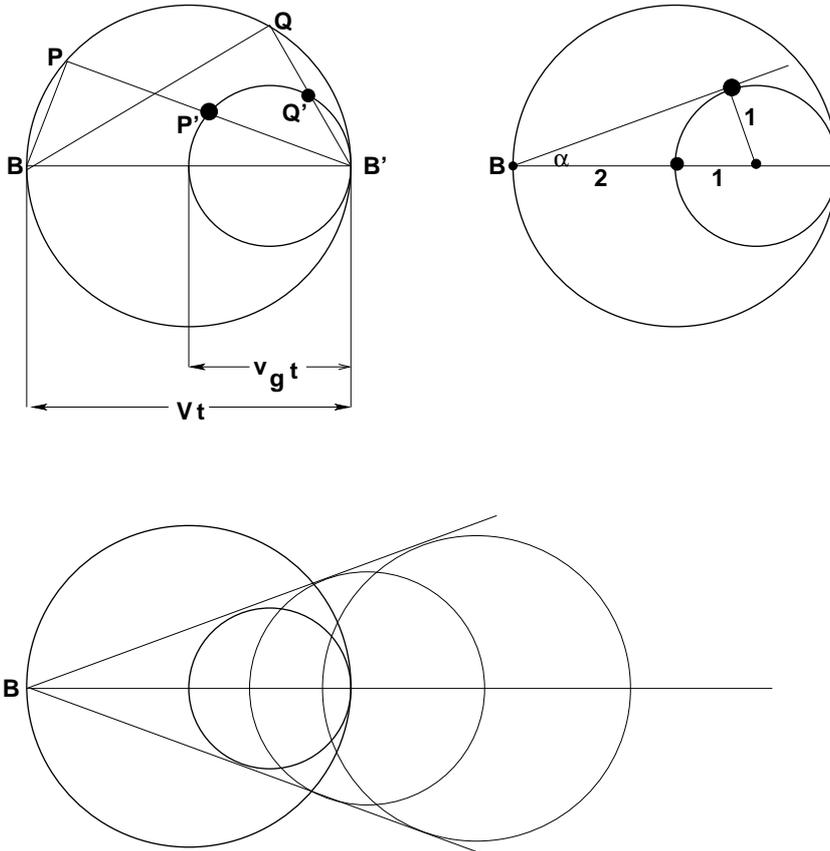
Figure 5: UPPER LEFT: Waves generated at the initial boat postion B' would reach a circle of diameter $BB' = Vt$, where B is the current boat position, if they traveled at the phase velocity $v_p = V \cos \theta$. In reality, they travel exactly half this distance, forming the inner circle. UPPER RIGHT: The half opening angle satisfies $\sin \alpha = 1/3$, for any choice of time t. LOWER: The ensemble of all such "inner circles" forms the wake of the boat.

or $\alpha = 19.5°$, a universal result for both ducks and battleships! Since our choice of t is arbitrary, there are an infinity of such smaller circles from a continuum of B's, and the circle ensemble fills the wedge behind the boat and forms a wake, as shown in the final drawing of figure 5.

Notice that this elegant little result is a consequence of nothing more than dimensional analysis and the concept of group velocity. The frequency of surface waves on deep water depends only upon the gravitational constant g and the wavenumber, and it follows from this alone that $\omega \propto k^{1/2}$. Thus $v_p = 2v_g$, and our entire analysis flows from this! Indeed, it is possible to do much more, like calculate the actual curved shape of the wave crests themselves (see Lighthill, pp. 276-7).

Consider next the group velocity of internal gravity waves. Calculating the components $\partial\omega/\partial k_x$ and $\partial\omega/\partial k_z$ from (176), one finds that they are the components of a vector orthogonal to the wave number $\boldsymbol{k}$! (Do it.) In other words, the group velocity points along the same direction as the surfaces of constant phase, which must also lie orthogonal to the wavenumber. This is very different from a sound wave, which propagates at right angles to surfaces of constant phase. An internal wave packet appears to be moving "sideways" relative to the undulations that form the wave surfaces.

Our final example is taken from capillary waves. The deep water group velocity is, from equation (144) or (145),

$$\frac{d\omega}{dk} = \frac{g + 3Tk^2/\rho}{2(gk + Tk^3/\rho)^{1/2}} \qquad (218)$$

Notice that as $k \to 0$, the group velocity of the waves increases (these are now surface gravity waves), and the same is true for $k \to \infty$ (these are true capillary waves). This means that the group velocity has a minimum at a well-defined wavenumber. In the neighborhood of this minimum, the waves behave nondispersively.

The interplay between phase and group velocities for capillary and gravity waves is nicely illustrated by water flowing over an obstacle in a stream. If the fluid velocity is smaller than the minimum *phase* velocity

$$c_m = (4gT/\rho)^{1/4}$$

(about 23 cm s^{-1}, see equation [145]), no wavenumbers can be stationary in the flow. But at flow velocities in excess of c_m, there are *two* solutions of the dispersion relation that are possible. One is a gravity wave, the other a capillary wave. Consider stationary disturbances caused by the obstacle, say a stone. In this case, the stream velocity is exactly equal to the phase velocity. The capillary wave has a group velocity in excess of its phase velocity, hence

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the disturbance is propagated back upstream, opposite to the current. (The stream is moving "subsonically" relative to this group velocity!) On the other hand, the gravity wave has a group velocity *less* than the stream velocity. Hence, this disturbance has no choice but to be carried downstream from the obstacle. Therefore the capillary and gravity waves are separated. This effect is clearly visible if you look carefully at stones in a stream. (A relatively slow current and small stones work best.) You will see small wavelength capillary waves piling up in front of the stones, and long wavelength gravitational surface waves trailing behind.

4.7   Wave Energy

A mechanical vibration excites sound waves in the air. These waves propagate through the medium, and excite other mechanical vibrations in the receiver. The energy necessary to excite these secondary oscillations must come from the waves themselves, i.e., sound waves transport energy. This argument is valid for any source and receiver, so in general we expect any type of propagating wave to transport energy.

Calculating the wave energy flux, a quantity second order (i.e., quadratic) in the amplitudes, requires some care. Our formal expression for the energy flux of a gaseous fluid is

$$\boldsymbol{\mathcal{F}} = \left(\frac{v^2}{2} + \Phi \right) \rho \boldsymbol{v} + \frac{\gamma}{\gamma - 1} P \boldsymbol{v} \tag{219}$$

Now waves have an oscillatory time and space dependence, so that if $\boldsymbol{v}$ is present only because of the wave itself, the kinetic energy flux will be of *third* order in the amplitude δv, and negligibly small. The term involving the potential Φ is proportional to the mass flux $\rho \boldsymbol{v}$, and corresponds to bulk motion of the fluid through the potential. In general, adiabatic waves cause no such bulk motions. (Imagine simple compression waves propagating through masses connected to one another by springs.)

This leaves the pressure term. Consider the following (specious) reasoning. The velocity is linear order in $\delta \boldsymbol{v}$. If P_0 is the equilibrium pressure, then $P_0 \delta \boldsymbol{v}$ vanishes on average, but $\delta P \delta \boldsymbol{v}$ does not. Therefore the wave energy flux is

$$\boldsymbol{\mathcal{F}} = \frac{\gamma \delta P \delta \boldsymbol{v}}{\gamma - 1} \; ?? \tag{220}$$

This, alas, is incorrect.

The reason that this approach fails is that it does not separate the component of the energy flux that is proportional to the mass flow from the

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component that is present when the mass flux vanishes. More specifically, we may write

$$\frac{\gamma}{\gamma - 1} P \boldsymbol{v} = \frac{\gamma}{\gamma - 1} c^2 \rho \boldsymbol{v} \tag{221}$$

where c^2 is $k_B T/m$, the product of the Boltzmann factor and temperature divided by the mass per particle. *Only the piece of δP that causes temperature fluctuations ($\rho \, \delta c^2$) results in true wave energy transport.* The sum of the correlated fluctuations in the product $\rho \boldsymbol{v}$ combine to give zero! This is just our claim that the mass flux from the wave vanishes, but the mathematical details are a little subtle, and we return to this point at the end. The true wave energy flux is therefore

$$\frac{\gamma}{\gamma - 1} \rho \, \delta c^2 \, \delta \boldsymbol{v} = \frac{\gamma}{\gamma - 1} P \left(\frac{\delta T}{T} \right) \delta \boldsymbol{v} \tag{222}$$

But the adiabatic gas law combined with the ideal gas equation of state gives

$$\frac{\gamma}{\gamma - 1} \left(\frac{\delta T}{T} \right) = \frac{\delta P}{P} \tag{223}$$

and we are left with a flux of

$$\boldsymbol{\mathcal{F}_w} = \delta P \, \delta \boldsymbol{v} \tag{224}$$

Notice that energy flux is just the rate at which the wave pressure does work on unit area of the fluid, a very satisfying and sensible result.

(Incidentally, I need hardly remind the reader that when a quantity like $\delta P \, \delta v$ is evaluated, one must take the real parts of δP and δv before multiplying them together! The simplifying trick of using complex variables works only in a linear calculation, when the real and imaginary parts of all quantities separate.)

What of our claim that the mass flux $\rho \boldsymbol{v}$ vanishes? This is reasonable physically, but formally appears to be in trouble, since $\delta \rho \, \delta \boldsymbol{v}$ does not, in fact, vanish for sound waves. Here, we must at last confront the difficulty that we are trying to calculate a quantity that is second order in the amplitudes (the energy flux) with a wave theory that has been calculated only to first order in the amplitudes! The true velocity in a wave is

$$\boldsymbol{v} = \boldsymbol{\delta v} + \boldsymbol{v}^{(2)} \tag{225}$$

where $\boldsymbol{v}^{(2)}$ includes all contributions of second order and higher (nonlinear) that are not calculated in first order (linear) theory. The true second order mass flux quantity is then

$$\rho \boldsymbol{v} = \delta \rho \, \boldsymbol{\delta v} + \rho_0 \boldsymbol{v}^{(2)} \tag{226}$$

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where ρ_0 is equilibrium mass density. It is this *sum* that vanishes, in the process allowing $\boldsymbol{v^{(2)}}$ to be determined (at least up to second order in the amplitudes).

To be thorough, we should expand everything to *second order* in the perturbation amplitudes, and retain all expressions through this order. In fact, what one finds is exactly what we found by the simple procedure of setting the mass flux equal to zero everywhere. When we investigate nonlinear one-dimensional sound wave theory, we will, in fact, calculate the second order velocity explicitly.

For water waves, life is simple. With ρ taken to be constant, the term proportional to $\boldsymbol{\nabla{\cdot}v}$ vanishes in our mechanical energy equation, there is no $\delta\rho$, and the energy flux term is always just $\delta P\,\delta\boldsymbol{v}$. The wave energy flux thus takes exactly the same form for adiabatic gases and for constant density liquids. It is always the rate at which wave pressure does work on unit area of the fluid, whether the fluid is a gas or a liquid.

Exercise. The intensity of sound is measured in decibels. If F_E is the energy flux of a sound wave, then

$$120 + 10\log_{10}(F_E/\mathrm{W\ m^{-2}})$$

is the number of decibels (dB) associated with the sound. Here, F_E is measured in watts per square meter. (One watt is one joule per second.) Zero dB corresponds to the onset of hearing, 120 dB is extremely painful. What are the fluid displacements associated with each of these limits? Take $\rho = 1.2$ kg m^{-3}, $a = 350$ m s^{-1}, $\nu = 400$ Hz.

Exercise. Consider the wave form introduced at the end of §4.6 for studying group velocity and causality,

$$\zeta \equiv \exp[\gamma(t - x/v_g)]\ \exp[(ikx - i\omega t)].$$

In general the wave energy density will be of the form $C\zeta^2$, where C is a constant independent of space and time. Show that the energy density is, on average, $\rho_E = C\exp[2\gamma(t - x/v_g)]/2$.

Next, evaluate $H = \int_0^\infty \rho_E dx$, and note that it is finite! This is the total energy contained in the region $x > 0$. The quantity dH/dt is the rate of the change of energy in this region, and this is equal to the wave energy flux entering into the region at $x = 0$. (In one dimension, the energy flux has units of energy per time.) Show that when $\gamma \to 0$, the energy flux is just the product of the energy density and the group velocity v_g, i.e., that the energy propagates at the group velocity.

4.8 Nonlinear Acoustic Waves

Up to this point, our wave theory has included only terms linear in the amplitude. For the case of one-dimensional adiabatic flow, however, it is possible to solve the time-dependent flow *exactly* to all orders in the amplitudes. The result is interesting not only in its own right, but for the light it throws on how linear waves may develop nonlinear features. Surprisingly, it is not simply a matter of the wave amplitudes becoming large.

This problem was first solved by the extraordinary mathematician Bernhard Riemann, and it is therefore often referred to as "the Riemann problem."

4.8.1 Quasilinear Theory of Partial Differential Equations

Before we begin our atttack on the Riemann problem, we need to review the theory of first order partial differential equations. Actually, because of our use of the Lagrangian derivative in the fluid equations, we already have the tools that we need. Consider an equation of the form

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = g \tag{227}$$

where v and g are functions that depend upon x, t, and possibly f. Such equations, linear in the derivatives but possibily nonlinear in the other variables, are known as *quasilinear* partial differential equations. The function v clearly plays the role of a velocity. Indeed, our equation (227) is manifestly of the form of a Lagrangian derivative, and we may say that *traveling with the fluid, $df/dt = g$*. More precisely, along the path

$$\frac{dx}{dt} = v(x, t, f) \tag{228}$$

the function f satisfies

$$\frac{df}{dt} = g(x, t, f) \tag{229}$$

The first equation for dx is called the trajectory characteristic; the second, for df, is called the solution characteristic. Together, these are "the characteristic equations." Note that our example is quite general, since any first order quasilinear PDE can be expressed in this Lagrangian derivative form. The solution of this form of PDE may be reduced to the solution of coupled ordinary differential equations for the characteristics, together with appropriate boundary conditions.

Let us see how this works with a simple example. Assume that at $t = 0$, the function f is given by $F(x)$, our boundary condition. Let v be a given

constant, and for simplicity let $g = f$. Go to a particular point on the $t = 0$ axis, call it $x = x_0$. (See figure 6.) The trajectory characteristic is a path emerging from $x = x_0$ satisfying $dx/dt = v$, namely

$$x = x_0 + vt \qquad \text{(trajectory characteristic.)} \tag{230}$$

The solution characteristic is the solution to $df/dt = f$ taken along this path:

$$f = F(x_0) \exp(t) \tag{231}$$

Note that we insist upon writing $F(x_0)$ rather than $F(x)$, because the boundary condition requires us to use the value of x along the trajectory characteristic $x = x_0 + vt$ at $t = 0$. This particular value of x is of course x_0. At any point (x, t) along the trajectory beginning at $x = x_0$, x_0 itself is given by $x_0 = x - vt$. Our final solution thus takes the form

$$f(x, t) = F(x - vt) \exp(t) \tag{232}$$

Direct substitution into the original PDE shows that this is indeed the solution.

4.8.2 The Steepening of Acoustic Waves

With the technique of the last section understood, we may proceed to the more complex problem of the coupled nonlinear fluid equations for one-dimensional acoustic disturbances.

Our problem is to solve the system of equations:

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{1}{\rho} \frac{\partial P}{\partial x} \tag{233}$$

$$\frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} = -\rho \frac{\partial v}{\partial x} \tag{234}$$

Equation (233) is the equation of motion; (234) is simple mass conservation. Our gas all lies on the same adiabat, $P = K\rho^\gamma$. Hence,

$$\frac{dP}{\rho} = d\left(\frac{a^2}{\gamma - 1}\right) \tag{235}$$

where a is the adiabatic sound speed

$$a^2 = \frac{\gamma P}{\rho} = \gamma K \rho^{\gamma - 1} \tag{236}$$

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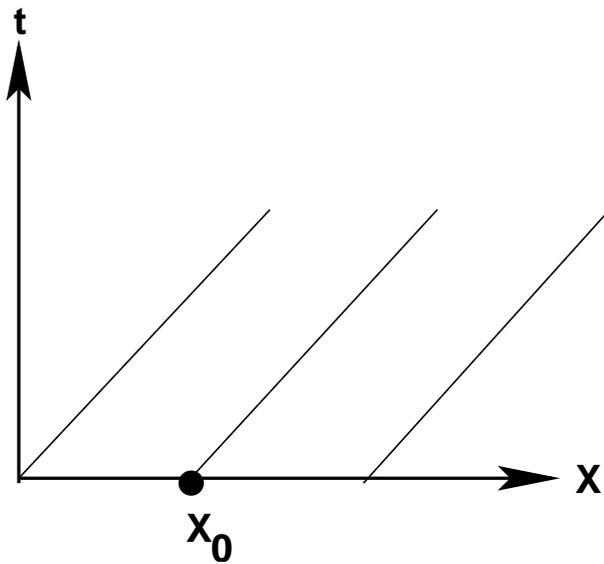Figure 6: Trajectory characteristics in the x, t plane. The point x_0 denotes the starting point of each trajectory. (Only one is shown explicitly in the figure.) The initial data are given as some function $f(x, t = 0)$ which we denote as $F(x_0)$. The solution characteristic equation then tells us how f changes along each of the trajectory characteristics, starting with the first point, x_0.

We express ρ and P in terms of a, and write our coupled equations in terms of v and a:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{2a}{\gamma - 1}\frac{\partial a}{\partial x} \tag{237}$$

$$\frac{2}{\gamma - 1}\left(\frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x}\right) = -a\frac{\partial v}{\partial x} \tag{238}$$

The next step is "obvious" — if you are Riemann. You add (237) and (238) to obtain

$$\frac{\partial}{\partial t}\left[v + \frac{2a}{\gamma - 1}\right] + (v + a)\frac{\partial}{\partial x}\left[v + \frac{2a}{\gamma - 1}\right] = 0. \tag{239}$$

Then you subtract the same two equations:

$$\frac{\partial}{\partial t}\left[v - \frac{2a}{\gamma - 1}\right] + (v - a)\frac{\partial}{\partial x}\left[v - \frac{2a}{\gamma - 1}\right] = 0. \tag{240}$$

This wonderful trick has decoupled the two equations into two quasi-linear equations that can be solved separately, and then their results can be combined. Equation (239) has the solution

$$\mathcal{R}_+ \equiv v + \frac{2a}{\gamma - 1} = \text{const} \quad \text{on} \quad \frac{dx}{dt} = v + a \tag{241}$$

the so-called $+$ characteristics, and equation (240) has the solution

$$\mathcal{R}_- \equiv v - \frac{2a}{\gamma - 1} = \text{const} \quad \text{on} \quad \frac{dx}{dt} = v - a \tag{242}$$

the $-$ characteristics. The $+$ and $-$ trajectory characteristics have the physical interpretation of sound waves moving relative to the fluid either forwards or backwards. Different quantities, $\mathcal{R}_+$ and $\mathcal{R}_-$, are maintained at constant values along these sound tracks. The $+$ and $-$ characteristics can, indeed *must*, cross each other without inconsistency. That is how we can solve independently for v and a! At a point where a $+$ and $-$ characteristic cross,

$$v = \frac{1}{2}(\mathcal{R}_+ + \mathcal{R}_-), \quad a = \frac{\gamma - 1}{4}(\mathcal{R}_+ - \mathcal{R}_-) \tag{243}$$

The true problem arises when two characteristics of the same type cross. This is when shocks form, the ultimate nonlinear behavior.

4.8.3 Piston Driven into Gas Cylinder

To understand in more detail how this happens, consider the classic problem of a piston moving in some prescribed manner at the head of a gas cylinder. The gas occupies the region $x > 0$. For $t < 0$, a piston at $x = 0$ is stationary. At $t = 0$ it begins to move with a velocity $v_P(t)$. This must also be the velocity of the gas immediately adjacent to the piston wall. The position of the piston is $x_P(t)$, the integral of $v_P(t)$ from $t = 0$ to time t. Refer to figure (7), the time-space diagram, in what follows.

The $-$ characteristics all originate from $t = -\infty$ and $x = +\infty$, hence

$$\mathcal{R}_- = v - \frac{2a}{\gamma - 1} = -\frac{2a_0}{\gamma - 1} \tag{244}$$

everywhere, where a_0 is the initial value of the sound speed in the undisturbed gas. This is the same constant on every $-$ characteristic, and so this equation is true throughout the flow: in a well-defined flow, every space-time point is linked to *some* $-$ characteristic emerging from the distant past. (We should always think of advancing with $dt > 0$ along any characteristic, since this is propagation of information. A $-$ characteristic emerging from the piston in the $dt > 0$ direction would head in the direction *behind* the piston! This is nonsense.)

The $+$ characteristics originate from the piston head, "moving" into the fluid ahead. For $t \leq 0$ the piston location corresponds to $x = 0$. For $t > 0$, this corresponds to the path $t_P = t_P(x_P)$, the inverse function $x_P(t_P)$. (Note that we explicitly label the piston path with subscript P.) In the lower half plane $t \leq 0$, the $+$ characteristics are straight lines

$$\frac{dx}{dt} = a_0, \quad x = a_0(t - t_0), \quad \mathcal{R}_+ = \frac{2a_0}{\gamma - 1} \quad (t \leq 0) \tag{245}$$

where t_0 is the time along the axis $x = 0$ at which a characteristic begins. The last characteristic emerging from the stationary piston comes from the point $x = t = 0$. Then, for $t > 0$, there is a nonvanishing velocity at the beginning of each $+$ characteristic (the piston head), v_P, which is a given function of time, t_P. To calculate how this changes the $+$ characteristics, let us first go back to the $-$ characteristics. The $-$ characteristics must penetrate everywhere that our solution is well-defined. Therefore, by the constancy of R_-, the sound speed at the piston head a_P is linked to its value at $-\infty$, and is determined by

$$v_P - \frac{2a_P}{\gamma - 1} \equiv \mathcal{R}_- = -\frac{2a_0}{\gamma - 1}. \tag{246}$$
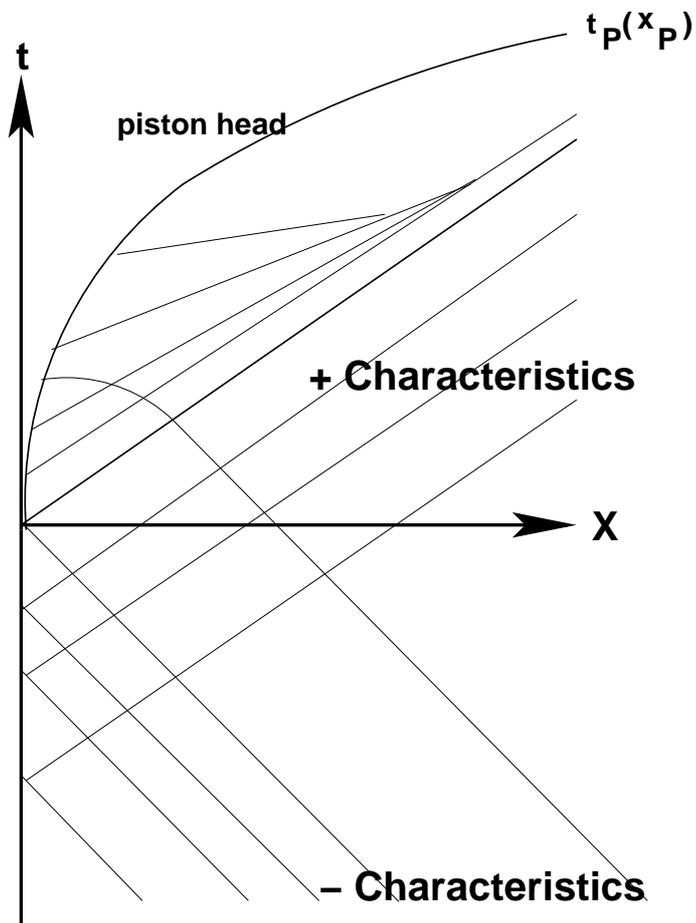
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Figure 7: Piston driven into a cylinder. The path of the piston is shown in the tx plane by the thick line. The thin lines are − characteristics and + characteristics. The latter always emerge from the cylinder along straight lines, but the slope becomes more shallow as the piston accelerates. Eventually the + characteristics cross and a shock must form.

If our problem is well-posed, this would have to be true for *any* v_p, a_P along the path of the piston. We may solve easily now for a_P interms of v_P:

$$a_P = a_0 + \frac{\gamma - 1}{2} v_P. \tag{247}$$

Now, for $t > 0$, at some arbitrary point in the flow Q

$$\mathcal{R}_+ = v + \frac{2a}{\gamma - 1} = v_P + \frac{2a_P}{\gamma - 1}, \tag{248}$$

where now the "P" refers to the point at the piston head connected with point Q in the flow by a $+$ characteristic. But, at the same arbitrary point Q in the flow,

$$v - \frac{2a}{\gamma - 1} = -\frac{2a_0}{\gamma - 1} = v_P - \frac{2a_P}{\gamma - 1}, \tag{249}$$

where once again, the "P" refers to the point at the piston at the head of the $+$ characteristic. This follows because while the point Q in the flow and the point P on the piston each lie on their own separate $-$ characteristic, along *any* $-$ characteristic, R_- is the same number! That number is always given by equation (246) for every point P along the piston. We thus find that v and a are both constant *along the $+$ characteristic joining P and Q*:

$$v = v_P = v_P(t_P) \quad \text{(a given function of } t_P) \tag{250}$$

$$a = a_P = a_0 + \frac{\gamma - 1}{2} v_P. \tag{251}$$

Note that this does not mean that v and a are constant throughout the region $t > 0$! Their value changes from one $+$ characteristic to another because a_P and v_P change from once $+$ characteristic to another. But since v and a are both constant along a $+$ characteristic, the equation of the $+$ trajectory must be a straight line:

$$\frac{dx}{dt} = v + a = v_P + a_P \rightarrow x - x_P = (v_P + a_P)(t - t_P), \tag{252}$$

or

$$x - x_P = [v_P(\gamma + 1)/2 + a_0](t - t_P). \tag{253}$$

With x_P and v_P given functions of t_P, equation (253) becomes an implicit equation for $t_P = t_P(x, t)$, and our problem is formally solved by equations (250) and (251) with the replacement of t_P by $t_P(x, t)$.

An explicit solution can be found for the case

$$v_P = \alpha t_P, \quad x_P = \alpha t_P^2/2, \tag{254}$$

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and piston that is accelerating into the cylinder. In this case (253) is a simple quadratic equation in t_P. This task we shall leave for the reader. But it is easy to see that something will go terribly wrong! The slope for each of the $+$ characteristics emerging at later and later times becomes steeper and steeper if one considers dx/dt, but less and less steep in the tx plane (i.e., dt/dx) shown in the figure. These characteristics soon cross, and the theory breaks down. This corresponds to the formation of a shock wave, as the density (or velocity) profile becomes steeper and steeper (characteristics becoming closer and closer). The point at which the characteristics cross corresponds to an infinite spatial gradient. Something new enters the problem: viscosity becomes crucial for resolving the shock profile.

Exercise. What happens mathematically in the analytic solution when the characteristics cross?

4.8.4 Driven Acoustic Modes

This section constains optional advanced material, included for your interest.

We turn next to the case of only slightly nonlinear acoustic waves. Let the piston position as a function of time be

$$x_P(t_P) = \epsilon \sin(\omega t_P) \tag{255}$$

where ϵ is a small, but finite constant parameter. Then

$$v_P = \epsilon \omega \cos(\omega t_P) \tag{256}$$

and

$$x = \epsilon \sin(\omega t_P) + [\epsilon \omega \cos(\omega t_P)(\gamma + 1)/2 + a_0](t - t_P) \tag{257}$$

This is too complicated to solve for t_P explicitly as a function of x and t for any ϵ, but it can be solved order-by-order. First, with $\epsilon = 0$,

$$t_P = t - \frac{x}{a_0} \tag{258}$$

which is just the standard "retarded time." Given a location x, a time t, and an unperturbed sound speed a_0, the above t_P is just the time at which the signal now at (x, t) originated—in *linear* theory. The velocity is then

$$v = v_P = \epsilon \omega \cos(\omega t - kx) \tag{259}$$

where the wavenumber is $k = \omega/a_0$. This is a standard linear plane wave. Finally the sound speed solution may be written

$$\frac{a - a_0}{a_0} = \epsilon \left(\frac{\gamma - 1}{2a_0} \right) \omega \cos(\omega t - kx) \tag{260}$$

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which again is a result in agreement with linear theory. Motion of the piston at some time t_P is simply mimicked at location x a time $t_P + x/a_0$ later.

We now enter the nonlinear regime. In equation (257), we obtain the first order correction to t_P by replacing t_P with its zeroth order expression $t - x/a_0$ in every term linear in ϵ, which we shall henceforth denote t_{P0}:

$$x = \epsilon \sin(\omega t_{P0}) + [\epsilon \omega \cos(\omega t_{P0})(\gamma + 1)/2 + a_0](t - t_P) \qquad (261)$$

The first order solution for t_P, which we shall call t_{P1} is then given by

$$t_{P1} = t - \frac{x}{a_0}\left(1 - \epsilon k \frac{\gamma + 1}{2}\cos(\omega t_{P0})\right) + \frac{\epsilon}{a_0}\sin(\omega t_{P0}) \qquad (262)$$

The corrected velocity is now just proportional to $\cos(\omega t_{P1})$. To this order, we see that there are periodic changes both to the explicit phase of the wave, as well as to the wavenumber.

Let us investigate how these changes affect the possibility of a systematic, nonvanishing velocity field at this higher order. Working *strictly* to first order in ϵ (which assumes, for example, that quantities such as $k^2 x \epsilon$ are small), repeated expansion of the cosine and sine functions gives

$$\cos(\omega t_{P1}) = \cos(\omega t - kx) - \frac{\epsilon(\gamma + 1)k^2 x}{4}\sin(2(\omega t - kx)) - \epsilon k \sin^2(\omega t - kx) \qquad (263)$$

Thus the average of $\cos(\omega t_{P1})$ does *not* vanish:

$$\langle \cos(\omega t_{P1})\rangle = -\frac{\epsilon k}{2} \qquad (264)$$

The velocity is $\epsilon \omega \cos(\omega t_{P1})$, so the nonvanishing average velocity is second order in ϵ.

We are now in a position to address the question of whether there is any mass flux associated with the sinusoidal oscillation of the piston, at least to the lowest order needed to do the calculation. The density is proportional to the $2/(\gamma - 1)$ power of the sound speed $a = a_P$,

$$\rho \propto a_P^{2/(\gamma-1)} \qquad (265)$$

Using our explict solution, we obtain

$$a_P^{2/(\gamma-1)} = a_0^{2/(\gamma-1)}\left(1 + (\epsilon k)\frac{\gamma - 1}{2}\cos(\omega t_P)\right)^{2/(\gamma-1)} \qquad (266)$$

(We do not yet specify the order of t_P.) The mass flux is proportional to

$$va^{2/(\gamma-1)} = \epsilon\omega\cos(\omega t_p)\, a_0^{2/(\gamma-1)}\left(1 + \epsilon k\cos(\omega t_P) + ...\right) \qquad (267)$$

Its average is computed as

$$\langle va^{2/(\gamma-1)}\rangle = a_0^{2/(\gamma-1)}\left(\epsilon\omega\langle\cos(\omega t_P)\rangle + \epsilon^2\omega k\langle\cos^2(\omega t_P)\rangle + ...\right) \qquad (268)$$

In the first term on the right, we need to use t_{P1} so that the contributions through order ϵ^2 to the mass flux are included. The second term is already of quadratic order, and t_{P0} is acceptable. Using equation (264), both terms in the equation are of magnitude $\epsilon^2\omega k/2$, but they occur with opposite signs. Thus, the mass flux vanishes through order ϵ^2, just as we argued more informally in an earlier section.

Notice how the nonlinear structure has emerged in our calculation from a simple sinusoidal traveling wave. An explicit phase function has appeared which itself has the form of a traveling wave, and the wavenumber is also modulated. Indeed, the following argument suggests that in nonlinear theory an exact solution should show a continuous steepening of the wave as time goes on. The sound speed is faster in regions of higher density than in lower density, so the crests of a traveling wave should always be trying to overtake the evacuated troughs in front of them. This is indeed what happens. But how do we interpret the limit in which the gradient becomes infinite, i.e., the density and velocity profiles have discontinuities? These structures are called shock waves, and our the topic of our next section.

4.9   Shock Waves

4.9.1   Rankine-Hugoniot Relations

We have seen that a sound wave propagating in a medium tends to move slightly more rapidly in regions where it is compressed (and the temperature is higher) than in regions where it is not. The effect of this that the peaks of a sound wave are constantly overtaking the troughs, and the profile of the wave steepens with time. In a finite amount of time, which can be calculated by the methods discussed above, the slope of the density formally becomes infinite, and if we were to continue to believe our equations, the density becomes double-valued!

No such nonsense is observed in nature, of course. What happens instead is that a wave continues to steepen until the characteristic length scale over which the density changes becomes comparable to the collisional mean free
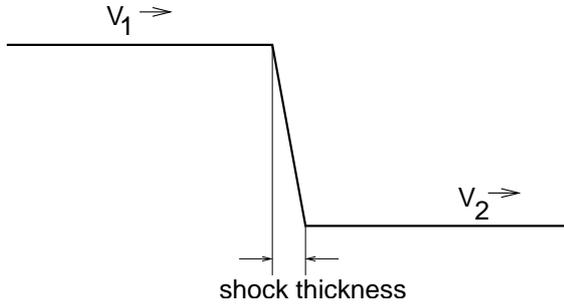
Figure 8: Schematic shock structure of the velocity field. Flow enter from the left ($x < 0$) at velocity v_1, passses through the shock transition ($x = 0$), slows, and emerges ($x > 0$) with velocity v_2. The transition layer is very narrow. The density, pressure and temperature profiles would all show sharp increases.

path of the gas particles. At this point, our simple adiabatic equations of motion certainly can no longer be trusted. This is precisely the scale at which viscosity and thermal conduction become important.

A description of how the disturbance behaves is as follows. The gas wave steepens until the gradient is, from a macroscopic point of view, a discontinuity in both the density and the velocity. This discontinuity is the shock wave. In reality, there is a continuous flow through the "discontinuity", and the fluid passes through the shock, making a rapid transition from one state of well-defined velocity, density, and pressure to another. But the transition is tightly regulated. Mass, momentum, and energy must all be conserved. This is sufficient to determine the final state of the gas, given its pre-shock state: the fate of the gas does not depend on the form of the viscosity or thermal conduction. The presence of dissipation, however, means that the entropy is not continuous across the shock. Rather, it increases irreversibly.

To be explicit, consider the fluid equations in the frame in which the shock is at rest at the location $x = 0$. The fluid enters from the side $x < 0$, makes a transition, and leaves at $x > 0$. Assume that the flow is entirely perpendicular to the plane of the shock, or *front*. (See figure 8.) The structure is steady in time. Under these conditions, the shock is well described by one-dimensional fluid equations. The equations of mass, momentum, and energy can all be written in the conservation form

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{F}} = 0 \tag{269}$$

where $\boldsymbol{\mathcal{F}}$ is an associated flux. For example, mass conservation is simply $\boldsymbol{\nabla} \cdot \rho \boldsymbol{v} = 0$. Sections 2.4 and 3.3 discuss the energy and momentum fluxes respectively. These fluxes apply to an adiabatic gas, whereas the shock front

is dissipative. We will discuss the dissipative equations later in the course; for now we note that the viscous and thermal conduction fluxes simply add directly to the adiabatic flux terms that are already present. These dissipative fluxes are negligible except in a very narrow layer within a few particle mean free paths of the front. In particular, equation (269) still holds in its conservation form for the mass, momentum and energy fluxes. If we now integrate (269) from one side of the shock to the other, beginning and ending well outside the dissipation zone, we find

$$\mathcal{F}_2 = \mathcal{F}_1 \tag{270}$$

where the labels 2 and 1 refer to the postshock ("downstream") and preshock ("upstream") regions. In zones 2 and 1, the adiabatic fluxes may be used. The equations of mass, momentum and energy conservation are therefore

$$\rho_2 v_2 = \rho_1 v_1 \tag{271}$$

$$P_2 + \rho_2 v_2^2 = P_1 + \rho_1 v_1^2 \tag{272}$$

$$\frac{\rho_2 v_2^3}{2} + \frac{\gamma P_2 v_2}{\gamma - 1} = \frac{\rho_1 v_1^3}{2} + \frac{\gamma P_1 v_1}{\gamma - 1} \tag{273}$$

The velocity v is the velocity normal to the shock front. These are three equations for three unknowns, P_2, ρ_2, and v_2. These unknowns can be solved for, once the upstream values P_1, ρ_1 and v_1 are specified. (Notice that any external potential forces are unimportant if the shock transition is narrow, since these forces are continuous from one side of the transition to the next.)

Eliminating ρ_2 and P_2 from (271–273), we find that the ratio $x = v_2/v_1$ satisfies the quadratic equation

$$x^2 - \frac{2(\gamma M^2 + 1)}{M^2(1 + \gamma)}x + \left(\frac{\gamma - 1 + 2/M^2}{\gamma + 1}\right) = 0 \tag{274}$$

where we define the so-called "Mach number" M by

$$M^2 = \frac{v_1^2}{\gamma P_1/\rho_1} = \frac{v_1^2}{a_1^2}. \tag{275}$$

This is the ratio of the upstream gas velocity to the upstream adiabatic sound speed, all squared. As usual, there are two solutions to this quadratic equation, one of which we know without doing any work at all: $x = 1$! (Obviously $v_1 = v_2$ satisfies the original equations.) But knowing that a quadratic equation of the form (274) has one root $x = 1$ means that the

other root is simply the constant term in the equation. This is the shock solution:

$$x = \frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma - 1 + 2/M^2}{\gamma + 1} \tag{276}$$

The pressure ratio is obtained from

$$P_2 = P_1 + \rho_1 v_1^2 - \rho_2 v_2^2 = P_1 + \rho_1 v_1 (v_1 - v_2)$$

which follows from (271) and (272), and one finds:

$$\frac{P_2}{P_1} = \frac{2\gamma M^2 + 1 - \gamma}{1 + \gamma}. \tag{277}$$

The temperature T is proportional to P/ρ, and works out to be

$$\frac{T_2}{T_1} = \frac{(2\gamma M^2 + 1 - \gamma)(\gamma - 1 + 2/M^2)}{(1 + \gamma)^2} \tag{278}$$

Equations (276-278) are collectively known as the *Rankine-Hugoniot* jump conditions for a plane parallel shock front.

4.9.2 Discussion

The study of shocks is rich and fascinating, and it could be an entire course by itself (see the two volumes of Courant and Friedrichs or Zel'dovich and Razier), so here we must be content to limit ourselves to just a few essential comments.

The R-H conditions become particularly simple in two limiting cases. The first is when $M^2 \to 1$, in which case all of our "2/1" ratios reduce to unity as well. The magnitude of the discontinuities of the density, pressure and temperatures is proportional to $M^2 - 1$, when this quantity is small. The existence of shocks requires an upstream velocity satisfying $M^2 > 1$, and a weak shock corresponds to the limit $M^2 \to 1$ from above. You should be able to show that the postshock Mach number is always less than unity, but that the entropy always increases across a shock front. Low entropy, ordered kinetic energy is converted into high entropy, disordered thermal energy.

The second limit of interest corresponds to a strong shock, $M^2 \to \infty$. Then the R-H conditions reduce to

$$\frac{v_2}{v_1} = \frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1}, \tag{279}$$

$$\frac{P_2}{P_1} = \frac{2\gamma M^2}{\gamma + 1} \tag{280}$$

$$\frac{T_2}{T_1} = \frac{2\gamma M^2(\gamma - 1)}{(1 + \gamma)^2} \tag{281}$$

The last equation may be rewritten as

$$T_2 = \frac{2(\gamma - 1)}{(\gamma + 1)^2} \frac{mv_1^2}{k} \tag{282}$$

where m is the mass per particle and k is the Boltzmann constant. The postshock temperature is independent of the preshock temperature in a strong shock, and instead depends only upon the kinetic energy per particle of the gas entering the shock.

Notice that although there is no limit to how much a shock can heat gas, there is a limit as to how much it can *compress* gas. For $\gamma = 7/5$ it is a factor of 6; for $\gamma = 5/3$ a factor 4. Indeed, it is precisely the fact that the gas is so strongly heated that limits the postshock density rise. The pressure becomes prohibitively high beyond density compression factors that are only of order unity or so, and further compression stops.

Strong shocks emerge when a large amount of energy is suddenly released in an exceedingly short period of time. We call this a "bomb." Much of the development of shock wave physics has been driven by the practical result that it is possible to determine the energy released in nuclear weapons by studying the structure of the flow behind the shock wave that is produced.

The velocity of the spherical shock wave produced by a bomb that has exploded in a uniform gas can depend only upon the energy released E and the initial density of the gas ρ. From these two quantities, dimensional analysis is sufficient to make a rough guess of what the shock radius r_{sh} and shock velocity v_{sh} are as a function of time t:

$$r_{sh} \sim (Et^2/\rho)^{1/5} \qquad v_{sh} \sim (2/5)(E/\rho t^3)^{1/5} \tag{283}$$

As it happens, the unknown proportionality constant here turns out to be very close to unity for values of γ of physical interest.

A small atomic bomb has a yield of about 10^{21} ergs (10^{14} J). The density of air at sea level is about 1.2 kg m^{-3}. Thus, 0.01 seconds after the blast, a shock wave with a velocity of roughly 4000 m s^{-1} propagates through the air. The speed of sound is only about 350 m s^{-1}, so this is a strong shock wave, $M^2 \simeq 130$. With an average mass per particle in the atmosphere of about 4.6×10^{-23} g, and $\gamma = 7/5$, the postshock temperature works out to be 7400 K. The air radiates approximately as a blackbody at this temperature,
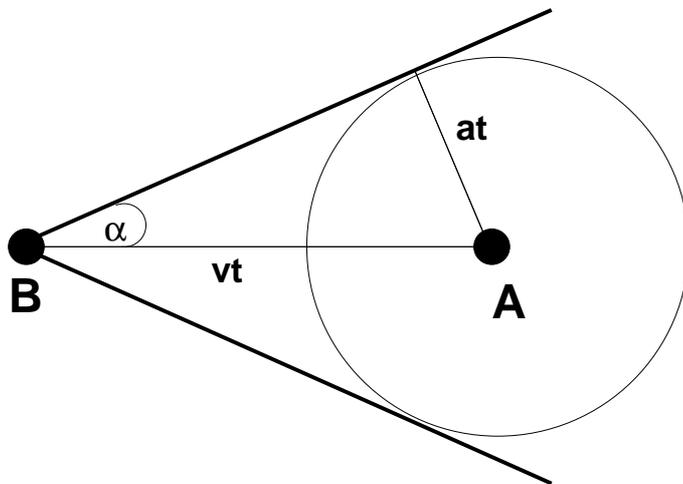
Figure 9: The Mach cone created by a supersonic aircraft. See text for details.

and would appear blue. (In fact, the gas becomes partially ionized and it is not quite fair to take a constant value of γ, but the numbers are not far off in the end.) Supernovae, nature's ultimate explosions, also produce strong shocks in the surrouding low density gaseous medium. These are discussed in my course in Astrophysical Gasdynamics.

In less explosive circumstances, when a terrestrial aircraft flies faster than the local speed of sound, a cone forms beyond the plane whose surface corresponds to a weak shock wave (see figure 9). In front of the cone, the air is completely undisturbed, since no acoustical signal can move faster than the plane. The interior of the cone marks the region in which sound waves caused by the plane have been able to propagate.

Following the figure, at time zero, the plane is located at A, and at time t it has traveled a distance Vt to point B. An acoustic disturbance has traveled a distance at during this time, and its sphere of influence lies behind the plane as shown. The ensemble of all such spheres fills the interior of the Mach cone with half opening-angle α such that

$$\sin \alpha = \frac{a}{V} = \frac{1}{M} \qquad (284)$$

By way of contrast, a subsonic aircraft remains always within the spheres of influence created by its acoustic disturbances. Lighthill makes the amusing observation that if we could only hear the conversation in a supersonic plane as it passed overhead, all the sounds would come out backwards: "pap pep

pip pop pup" becomes "pup pop pip pep pap"! This can perhaps be thought of as a negative Doppler shift.

Notice that the condition (284) states that $V \sin \alpha = a$. This states that relative to the plane, sound waves have a velocity component normal to the cone of $a - V \sin \alpha = 0$, i.e., they are stationary. This is why the Mach cone surface is itself stationary, and why a BOOM is heard as the cone passes over head. All the acoustic disturbances generated by the plane pile up along the surface of the cone and are perceived at once, whereas in the cone interior they are spread out and spatially dispersed.

The other consequence of the condition (284) is that the component of the gas velocity normal to the cone's surface corresponds to a Mach number of exactly unity: $V \sin \alpha / a = 1$, regardless of the plane's Mach number through air, $V/a > 1$. The faster the plane travels, the sharper the Mach cone becomes, so that the normal component of the gas velocity always maintains a Mach number of unity. This means that the conic surface corresponds to a weak shock. (The fact that shock is weak also means that it does not matter whether we use the pre- or post-shock value of a.) This is a good thing for human beings: the over-pressure behind a moderately strong $M^2 = 2$ shock in the atmosphere would be the same as being 10 meters under water! That *would* be quite a BOOM.

A well-designed supersonic aircraft should have a nose that tapers to a sharp point, so that the apex of the cone is not blunt, and the $M = 1$ conical surface can extend all the way to the tip. A blunt nose would result in a *strong shock* with all the adverse temperature and pressure increases that entails. This is why the Concorde looks the way it does: a sharp tip and compact swept-back wings to stay within its Mach cone. Subsonic aircraft, on the other hand, are designed with a smooth nose, so that the flow streamlines can smoothly pass over the airplane without becoming turbulent, and straight extended wings, so that the normal component of the air speed is large and can generate lift.

4.10   Stable Nonlinear Waves on Water

4.10.1   Rayleigh's Solitary Wave

Recall the dispersion relation for water waves in a body of depth H:

$$\omega^2 = gk \tanh(kH) \tag{285}$$

When kH is small,

$$\omega^2 = gk[(kH) - (kH)^3/3 + ...] \tag{286}$$

This implies a phase velocity $c = \omega/k$ of

$$c = c_0(1 - (kH)^2/6 + ...)$$ (287)

where $c_0^2 = gH$. This quadratic departure from a constant phase velocity is typical not just of water waves, but any system which is weakly dispersive. (Departures from constant c_0 are of order k^2 if the dispersion relation is smooth and $\pm k$ symmetric.) The simplest possible wave equation that leads to this phase velocity (keeping only the quadratic term in kH) is:

$$\frac{\partial v}{\partial t} + c_0 \frac{\partial v}{\partial x} + \frac{c_0 H^2}{6} \frac{\partial^3 v}{\partial x^3} = 0$$ (288)

A considerably more detailed analysis of *nonlinear* shallow water waves, which we shall pursue in the next section, yields an equation that is hardly different: the velocity v is added to the wave speed c_0 in the second, convective, term:

$$\frac{\partial v}{\partial t} + (c_0 + v) \frac{\partial v}{\partial x} + \frac{c_0 H^2}{6} \frac{\partial^3 v}{\partial x^3} = 0,$$ (289)

and results in the only nonlinear term in the equation. This is one of several different but equivalent forms of the *Korteweg-de Vries* (KdV) equation, which is generic to nonlinear, slightly dispersive wave systems. The best known applications are to water and plasma waves.

In the case of water waves, the velocity v is not the vertical displacement velocity, but is instead proportional to the excess wave speed beyond $c_0 = \sqrt{gH}$ of linear theory. More precisely, let $c = \sqrt{gh}$, be the nonlinear wave speed, where h includes both the undisturbed depth H plus the wave height. Then, it can be shown that c_0 in the second term in the linear wave equation (288) is replaced by

$$c_0 + v = 3c - 2c_0$$

in nonlinear theory. Hence, we identify $v = 3(c - c_0)$. (See Acheson pp. 89–92, and the problem at the end of the next section.)

The KdV equation is famous for the fact that it has an exact solution, first discovered by Rayleigh.[2] Let $v = f(x - Ut)$, where U is a constant velocity to be determined. We seek a solution in which the function $f(X)$ vanishes at large $|X|$, an isolated traveling wave form that retains its profile as it propagates. If we substitute our expression for v into the above equation, there obtains

$$(c_0 - U)f' + f'f + \frac{c_0 H^2}{6} f''' = 0$$ (290)

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[2] Actually, Rayleigh discovered the solution of the KdV equation before the equation was discovered! He derived his result directly from the fundamental fluid equations.

where f' is the derivative of f with respect to the argument $X = x - Ut$. The expression on the left is an exact derivative, and may be immediately integrated to give

$$(c_0 - U)f + \frac{f^2}{2} + \frac{c_0 H^2}{6} f'' = 0 \qquad (291)$$

where the boundary conditions at infinity have been used to set the integration constant equal to 0. Multiplication by f' gives us another exact derivative on the left. Then, integration and implementaion of the boundary conditions leads to

$$(f')^2 = \frac{2}{c_0 H^2} f^2 \left[3 \left(U - c_0 \right) - f \right]. \qquad (292)$$

This is a separable first order equation:

$$\int \frac{df}{f \sqrt{3(U - c_0) - f}} = \pm \left(\frac{2}{c_0 H^2} \right)^{1/2} X. \qquad (293)$$

The easiest way to do the integral is to substitute $f = 3(U - c_0)\text{sech}^2 (s)$, and we find that

$$v(x,t) = f = 3(U - c_0)\text{sech}^2 \left[\sqrt{\frac{3}{2} \frac{(U - c_0)}{c_0 H^2}} (x - Ut) \right] \qquad (294)$$

This is the classical, (nearly) exact, nonlinear wave form in which the steepening caused by the finite amplitude is exactly balanced by dispersional spreading. The relationship between nonlinear wave speed $U - c_0$ and finite wave amplitude a follows from the finite amplitude wave speed

$$c^2 = gh, \qquad (295)$$

with $h = H + \delta h = H + a$. Then

$$ga \equiv g\delta h = 2c_0 \delta c = 2\sqrt{gH}(U - c_0), \qquad (296)$$

whence

$$U = c_0 + \frac{a}{2}\sqrt{\frac{g}{H}}. \qquad (297)$$

It follows that larger amplitude waves have larger velocities.

The solution (294) was first observed on the Union Canal at Hermiston, Scotland in 1834 by a talented Scottish engineer, John Scott Russell. His own description is both charming and informative:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

Exercise. With 1 mile equal to 1.6 km and 1 foot equal to 0.3 m, can you compute the depth H of the canal from Russell's data? Is the ~ 30 foot length of the disturbance consistent with our solution?

Solitary waves became very fashionable in particle physics when it was discovered that two such waves could interact, and even though each wave was nonlinear, they emerged from the "collision" completely unaltered! These were quickly dubbed *solitons*, and viewed as models of elementary particles. Particle physics has since moved on, but soliton theory (the name has become generic for solitary waves) remains an active area of research. In fiber optics, passage of a signal through a medium causes both nonlinear and dispersive effects. The fact than these effects can be made to mutually cancel allows signal transfer across vast distances with little distortion, a result of considerable practical benefit. A fiber optics cable linking Glasgow and Edinburgh runs beneath the original canal Russell used for his prescient observations of solitary waves (see www.ma.hw.ac.uk/solitons/press.html). If you are reading these notes online, you may in fact be making direct use of the KdV equation!

4.10.2   Derivation of the Korteweg-de Vries Equation

This section contains optional advanced material, included for your interest.

In this section, we give a rigorous derivation of the Korteweg-de Vries equation, based on the treatment of Drazin and Johnson (*Solitons: An Introduction*).

We seek two-dimensional, irrotational solutions to the equations of motion. The vertical direction is z, and the horizontal direction (the direction of wave propagation) is x. The depth of our channel is H and the displacement

of the surface is denoted as η. A typical horizontal scale (the "wavelength") is L, the ratio $\delta = H/L$ is small if the limit where the waves are not dispersive. If a is the maximum surface displacement, the ratio $\alpha = a/H$ is assumed to be small throughout this analysis. δ, we shall see, can be somewhat more general.

The velocity may be derived from a scalar potential $\boldsymbol{v} = \boldsymbol{\nabla}\phi$. Thus, mass conservation is

$$\boldsymbol{\nabla}{\cdot}\boldsymbol{v} = \nabla^2\phi = 0 \tag{298}$$

The equation of motion for a velocity field with vanishing vorticity is

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\left(\frac{v^2}{2}\right) = -\boldsymbol{\nabla}\left(\frac{P}{\rho} + \Upsilon\right) \tag{299}$$

where Υ is the gravitational potential energy. We will take the zero of Υ to be at $z = H$, so that $\Phi_g = g\eta$. We use the subscripts to denote partial differentiation,

$$\phi_x = \frac{\partial \phi}{\partial x}, \qquad \phi_z = \frac{\partial \phi}{\partial z}, \qquad \phi_t = \frac{\partial \phi}{\partial t}, \tag{300}$$

and the same with other variables.

The equation of motion may be integrated immediately,

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + \frac{P}{\rho} + g\eta = f(t) \tag{301}$$

where $f(t)$ is a function of time only. This function can be absorbed into the definition of ϕ without any physical consequence, so we may set $f(t) = 0$. (ϕ is defined only up to an arbitrary additive function of time.)

On the surface $z = H + \eta$, $P = 0$, and our dynamical equation becomes the boundary condition

$$\phi_t + \frac{1}{2}\left(\phi_x^2 + \phi_z^2\right) + g\eta = 0, \quad \text{on } z = H + \eta. \tag{302}$$

The other boundary condition on $z = H + \eta$ is that $v_z = d\eta/dt$, or

$$\phi_z = \eta_t + \phi_x\eta_x, \quad \text{on } z = H + \eta. \tag{303}$$

since the displacement η is a function of x and t. Our final equation is the Laplace equation for ϕ, together with the boundary condition that the vertical velocity ϕ_z vanishes at the bottom $z = 0$:

$$\phi_{xx} + \phi_{zz} = 0, \text{ with } \phi_z = 0 \text{ at } z = 0. \tag{304}$$

We seek solutions to equation (304) subject to the boundary condtions (302) and (303).

To make progress, we need to know the relative magnitudes of the terms in our equations. We first introduce the dimensionless forms of the x, z, and η variables. These are X, Z, and ζ, respectiely:

$$x = XL, \quad z = HZ, \quad \eta = a\zeta. \tag{305}$$

Recale that $\alpha = a/H$ is assumed to be small. No assumptions are made for ratio $\delta = H/L$. Our dimensionless time T is defined by

$$t = T(L/\sqrt{gH}) \tag{306}$$

i.e., T is the time computed in units of the time it takes a linear wave to cross a distance L. The basic scaling for ϕ comes from the assumption that the wave is driven by gravity and only slightly nonlinear, $\phi_t \sim \eta g$. In other words, the characteristic size for ϕ is

$$\phi_{char} \sim Tag \sim La\sqrt{\frac{g}{H}}, \tag{307}$$

hence we define the dimensionless potential Φ by

$$\phi = La\sqrt{\frac{g}{H}}\Phi \tag{308}$$

Partial derivative rescalings are therefore

$$\eta_t = \frac{a\sqrt{gH}}{L}\zeta_T, \quad \eta_x = \frac{a}{L}\zeta_X, \tag{309}$$

and

$$\phi_x = a\sqrt{\frac{g}{H}}\Phi_X, \quad \phi_z = aL\sqrt{\frac{g}{H^3}}\Phi_Z, \quad \phi_t = ag\Phi_T. \tag{310}$$

We may now formulate our problem in dimensionless variables. The Laplace equation is

$$\Phi_{ZZ} + \delta^2\Phi_{XX} = 0 \tag{311}$$

The dynamical boundary equation (302) is

$$\Phi_T + \frac{\alpha}{2}\left(\Phi_X^2 + \frac{1}{\delta^2}\Phi_Z^2\right) + \zeta = 0 \quad \text{on } Z = 1 + \alpha\zeta. \tag{312}$$

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and the kinematic surface boundary condition (303) is

$$\delta^2(\zeta_T + \alpha\zeta_X\Phi_X) = \Phi_Z \quad \text{on } Z = 1 + \alpha\zeta. \tag{313}$$

We now scale δ out of the equations, so that only the small parameter α remains. Let

$$\xi = (\alpha^{1/2}/\delta)[X - T], \quad \tau = (\alpha^{3/2}/\delta)T, \quad \psi = (\alpha^{1/2}/\delta)\Phi \tag{314}$$

Then

$$\frac{\partial}{\partial X} = \frac{\alpha^{1/2}}{\delta}\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial T} = \frac{\alpha^{3/2}}{\delta}\frac{\partial}{\partial \tau} - \frac{\alpha^{1/2}}{\delta}\frac{\partial}{\partial \xi} \tag{315}$$

Notice that the ξ variable is in essence a change of coordinates into a frame moving with at the velocity $\sqrt{gH}$ of the linear wave. Our equations become

$$\psi_{ZZ} + \alpha\psi_{\xi\xi} = 0, \quad \psi_Z = 0 \text{ at } Z = 0, \tag{316}$$

$$\alpha\psi_\tau - \psi_\xi + \zeta + \frac{1}{2}\left[\alpha(\psi_\xi)^2 + (\psi_Z)^2\right] = 0 \quad \text{on } Z = 1 + \alpha\zeta, \tag{317}$$

and

$$\alpha^2\zeta_\tau - \alpha\zeta_\xi + \alpha^2\zeta_\xi\psi_\xi = \psi_Z \quad \text{on } Z = 1 + \alpha\zeta. \tag{318}$$

As promised, δ has disappeared from the governing equations.

The remainder of our problem is a straightforward expansion in the small parameter α. Let

$$\psi = \psi_0 + \alpha\psi_1 + \alpha^2\psi_2 + ... \tag{319}$$

$$\zeta = \zeta_0 + \alpha\zeta_1 + \alpha^2\zeta_2 + ... \tag{320}$$

To leading order, our equations are therefore

$$\psi_{0ZZ} = 0, \quad \text{with } \psi_{0Z} = 0 \text{ at } Z = 0 \tag{321}$$

$$\zeta_0 - \psi_{0\xi} + (\psi_{0Z})^2/2 = 0 \quad \text{on } Z = 1, \tag{322}$$

and

$$\psi_{0Z} = 0 \quad \text{on } Z = 1. \tag{323}$$

Equation (321) implies $\psi_{0Z} = 0$ everywhere, so (323) is automatically satisfied. Hence

$$\psi_0 = \theta_0(\xi, \tau) \tag{324}$$

where θ_0 is an arbitrary function of ξ and τ, but not Z. Therefore, equation (322) becomes

$$\zeta_0 = \psi_{0\xi} = \theta_{0\xi}. \tag{325}$$

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At this stage, our ψ expansion is

$$\psi = \theta_0(\xi, \tau) + \alpha\psi_1 + \alpha^2\psi_2 + ..., \tag{326}$$

and thus

$$\psi_{ZZ} = \alpha\psi_{1ZZ} + \alpha^2\psi_{2ZZ} + ... \tag{327}$$

and

$$\psi_{\xi\xi} = \theta_{0\xi\xi} + \alpha\psi_{1\xi\xi} + \alpha^2\psi_{2\xi\xi} + ... \tag{328}$$

The ζ is expansion is

$$\zeta = \theta_{0\xi} + \alpha\zeta_1 + ... \tag{329}$$

The Laplace equation to order α reads

$$\psi_{1ZZ} = -\theta_{0\xi\xi} \tag{330}$$

Hence

$$\psi_{1Z} = -Z\theta_{0\xi\xi} \tag{331}$$

(since we must have $\psi_{1Z} = 0$ at $Z = 0$), and

$$\psi_1 = -\frac{Z^2}{2}\theta_{0\xi\xi} + \theta_1(\xi, \tau) \tag{332}$$

where now θ_1 is an arbitrary function of ξ and τ. Finally, the α^2 Laplace terms give

$$\psi_{2ZZ} = -\psi_{1\xi\xi} = \frac{Z^2}{2}\theta_{0\xi\xi\xi\xi} - \theta_{1\xi\xi} \tag{333}$$

Integrating with respect to Z twice as before, and using the $Z = 0$ boundary condition gives

$$\psi_{2Z} = \frac{Z^3}{6}\theta_{0\xi\xi\xi\xi} - Z\theta_{1\xi\xi} \tag{334}$$

and

$$\psi_2 = \frac{Z^4}{24}\theta_{0\xi\xi\xi\xi} - \frac{Z^2}{2}\theta_{1\xi\xi} + \theta_2(\xi, \tau) \tag{335}$$

where θ_2 is an arbitrary function of ξ and τ. The ψ expansion is now complete through order α^2:

$$\psi = \theta_0 + \alpha\left(\theta_1 - \frac{Z^2}{2}\theta_{0\xi\xi}\right) + \alpha^2\left(\theta_2 - \frac{Z^2}{2}\theta_{1\xi\xi} + \frac{Z^4}{24}\theta_{0\xi\xi\xi\xi}\right) + ... \tag{336}$$

We will need ψ_ξ through order α only,

$$\psi_\xi = \theta_{0\xi} + \alpha\left(\theta_{1\xi} - \frac{Z^2}{2}\theta_{0\xi\xi\xi}\right) + ..., \tag{337}$$

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but we require ψ_Z through order α^2:

$$\psi_Z = -\alpha Z \theta_{0\xi\xi} + \alpha^2 \left(-Z\theta_{1\xi\xi} + \frac{Z^3}{6}\theta_{0\xi\xi\xi\xi}\right) + ... \tag{338}$$

We return now to our surface equations (317) and (318). Through terms of order α, equation (317) reads

$$\alpha\theta_{0\tau} - \left[\theta_{0\xi} + \alpha(\theta_{1\xi} - \frac{Z^2}{2}\theta_{0\xi\xi})\right] + \theta_{0\xi} + \alpha\zeta_1 + \frac{\alpha^2}{2}\theta_{0\xi}^2 = 0 \tag{339}$$

or

$$\zeta_1 - \theta_{1\xi} = -\frac{1}{2}\left(\theta_{0\xi}^2 + \theta_{0\xi\xi}\right) - \theta_{0\tau} \tag{340}$$

(Note that in terms of order α, we may set $Z = 1$.) In equation (318) we expand through order α^2. This gives

$$\alpha^2\zeta_{0\tau} - \alpha\zeta_{0\xi} - \alpha^2\zeta_{1\xi} + \alpha^2\theta_{0\xi}\zeta_{0\xi} = -\alpha(1+\alpha\zeta_0)\theta_{0\xi\xi} - \alpha^2\left(\theta_{1\xi\xi} + \theta_{0\xi\xi\xi\xi}/6\right) \tag{341}$$

Terms of order α cancel out, and to order α^2 we find that

$$\zeta_{1\xi} - \theta_{1\xi\xi} = \zeta_{0\tau} + \theta_{0\xi}\zeta_{0\xi} + \zeta_0\theta_{0\xi\xi} - \theta_{0\xi\xi\xi\xi}/6 \tag{342}$$

Differentiating equation (340) with respect to ξ and setting it equal to equation (342) gives

$$-\frac{1}{2}\left(2\theta_{0\xi}\theta_{0\xi\xi} + \theta_{0\xi\xi\xi}\right) - \theta_{0\tau\xi} = \zeta_{0\tau} + \theta_{0\xi}\zeta_{0\xi} + \zeta_0\theta_{0\xi\xi} - \frac{\theta_{0\xi\xi\xi\xi}}{6} \tag{343}$$

If we now recall that $\theta_{0\xi} = \zeta_0$, we may simplify our result into a single equation for ζ_0,

$$2\zeta_{0\tau} + 3\zeta_0\zeta_{0\xi} + \frac{\zeta_{0\xi\xi\xi}}{3} = 0. \tag{344}$$

We may now drop the "0" subscript from our dimensionless amplitude ζ without ambiguity, and arrive at the Korteweg-de Vries equation in its standard form:

$$\zeta_\tau + \frac{3}{2}\zeta\zeta_\xi + \frac{\zeta_{\xi\xi\xi}}{6} = 0 \tag{345}$$

The KdV equation equation expresses a balance between the excess wave velocity beyond linear theory, nonlinear steepening, and linear dispersion, all of which are of the same order.

Exercise. Show that equations (289) and (345) are equivalent.

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5 Steady Irrotational Flow in Two Dimensions

A very broad class of flow depends only upon two spatial variables, the third axis being one of symmetry. The most important example is probably the flow around the wing of an airplane, which is, to a first approximation, independent of the position along the wing. (This approach obviously breaks down at the ends corresponding to the wing tip and the junction of the wing with the aircraft body.) If, in addition to two-dimensional symmetry, the flow also has the property of being *irrotational*, it is possible to apply very powerful complex variable techniques to the problem of finding explicit flow solutions.

An irrotational flow in a simply-connected domain is one in which any closed-loop line integral of the velocity $\boldsymbol{v}$ vanishes:

$$\int_C \boldsymbol{v} \cdot \boldsymbol{ds} = 0 \tag{346}$$

(Informally speaking, a simply-connected domain is one that has no holes.) Stokes' theorem applied to the line integral then gives

$$\int_A (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \boldsymbol{da} = 0 \tag{347}$$

where A is any area bounded by the curve C. Since the area is arbitrary, $\boldsymbol{\nabla} \times \boldsymbol{v}$ must vanish everywhere in the fluid. Notice that if the flow is singular the line integral need not vanish if it encompasses the singularity! Thus the flow $Rv_\phi = $ constant has a nonvanishing line integral if the path surrounds the singular origin, but a vanishing integral if the path does not surround the origin. In this chapter, therefore, we shall study two-dimensional velocity fields that locally satisfy the equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \qquad \text{incompressible flow} \tag{348}$$

and

$$\boldsymbol{\nabla} \times \boldsymbol{v} = 0 \qquad \text{irrotational flow} \tag{349}$$

5.1 The potential and stream functions

Incompressible flow in two dimensions must obey the equation

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{350}$$

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An obvious way to satisfy this equation is to introduce a function of Ψ with the property

$$v_x = \frac{\partial \Psi}{\partial y}, \qquad v_y = -\frac{\partial \Psi}{\partial x} \tag{351}$$

The divergence-free condition is then identically satisfied by the equality of mixed partial derivatives. This is a special case of the more general result that a divergence-free field can always be written as the curl of another field, the so-called vector potential. In the case of two-dimensional planar flow, only one component of the vector potential is needed (not three), and this Ψ field is known as the *stream function*. It gets its name from the fact that

$$v_x \frac{\partial \Psi}{\partial x} + v_y \frac{\partial \Psi}{\partial y} = 0, \tag{352}$$

i.e., curves of $\Psi = $ constant are parallel to the velocity streamlines.

But the velocity field is also irrotational, and so must be derivable from a scalar potential Φ:

$$v_x = \frac{\partial \Phi}{\partial x}, \qquad v_y = \frac{\partial \Phi}{\partial y} \tag{353}$$

The functions Ψ and Φ are more than two different functions that determine the velocity field; they are in fact deeply interrelated. Notice, for example, that

$$0 = v_x \frac{\partial \Psi}{\partial x} + v_y \frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial x}\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{\partial \Phi}{\partial y} \tag{354}$$

Thus, the gradients of Ψ and Φ are orthogonal, and it follows that lines of constant Ψ and lines of constant Φ are orthogonal. A constant Φ curve is parallel to the gradient of a constant Ψ curve, and vice-versa.

The relationships

$$v_x = \frac{\partial \Phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \qquad v_y = \frac{\partial \Phi}{\partial y} = -\frac{\partial \Psi}{\partial x} \tag{355}$$

will be recognized by all students of complex variable theory as the *Cauchy-Riemann* relations. They imply that there must exist an analytic function w of a complex $z = x + iy$ such that

$$w(z) = \Phi(x, y) + i\Psi(x, y). \tag{356}$$

In other words, Φ and Ψ are respectively the real and imaginary parts of an analytic complex function, $w(z)$.

The Cauchy-Riemann relations are easily proven. By definition, an analytic function $w(z)$ has a well-behaved derivative dw/dz. In particular,

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differentiating $w = w(x + iy)$ with respect to x must give the same result as differentiating with respect to iy. Hence

$$\frac{\partial \Phi}{\partial x} + i\frac{\partial \Psi}{\partial x} = -i\frac{\partial \Phi}{\partial y} + \frac{\partial \Psi}{\partial y} \tag{357}$$

Equating real and imaginary parts then gives the Cauchy-Riemann equations.

Once again, by equality of mixed derivatives, the Cauchy-Riemann relations imply that both Φ and Ψ each satisfy the Laplace equation,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \tag{358}$$

Exercise. What are the Cauchy-Riemann equations in polar coordinates $x = R\cos\phi, y = R\sin\phi$? Show that they lead to the Laplace equation:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0 \tag{359}$$

and similarly for Ψ.

Note that the flow velocity may be obtained very simply from the function $dw(z)/dz$:

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial \Phi}{\partial x} + i\frac{\partial \Psi}{\partial x} = v_x - iv_y \tag{360}$$

Hence

$$v = \sqrt{v_x^2 + v_y^2} = \left|\frac{dw}{dz}\right| \tag{361}$$

Conversely, if we know the velocity fields as functions of z, we may find the corresponding w by direct integration.

Exercise. Show that

$$e^{i\phi}\frac{dw}{dz} = v_R - iv_\phi$$

5.2 Flows in the Complex Plane

We are interested in solving for two-dimensional velocity fields in the xy plane in the presence of bounding surfaces. We have shown, quite remarkably, that this plane may be taken to be the complex $z = x + iy$ plane, and that our solution reduces to finding an analytic function $w(z)$ in this plane. To

find the solution for a flow with a bounding surface $g(x, y) =$ constant, we need to find some $w(z)$ whose stream function (imaginary part) $\Psi(x, y)$ is constant along this same surface. Here is the critical idea: let $Z = f(z)$ map a boundary curve in the x, y plane into some other boundary curve in the X, Y plane. To find the flow with this new boundary, we simply map *all* the other flow streamlines along with boundary, using the same $f(z)$. These new streamlines then define the flow with the new bounding surfaces.

Let us be mathematically precise: if $w(z)$ is the first flow solution with one set of boundaries, and $Z = f(z)$ maps these x, y boundary curves into X, Y boundary curves (which will generally have a different shape), then $w(F(Z))$ is the new flow $W(Z)$ in the Z plane. Here $z = F(Z)$, the inverse function of f. This is the powerful technique of conformal mapping, which we will study in more detail later. For now, let us study some specific examples to see how this idea works in practice.

5.2.1 Uniform Flows

Uniform flow over the x-axis is perhaps the simplest one can imagine. With $v_x = U$, $v_y = 0$, we have $w(z) = Uz$. If the flow is inclined at an angle α relative to the x-axis, $v_x = U \cos \alpha$, $v_y = U \sin \alpha$, $dw/dz = Ue^{-i\alpha}$, $w = Uze^{-i\alpha}$.

The function $w(z) = z$ could describe uniform flow toward positive x in the upper half plane with a $y = 0$ boundary. The transformation function $Z = f(z) = z^{1/2}$ maps the $x < 0$ axis $Re^{i\pi}$ (part of the boundary) into $iR^{1/2}$, the new Y axis (and part of the new boundary). The $x > 0$ axis is unchanged, mapping into $X > 0$. Therefore, the new flow $W(Z) = w(F(Z)) = Z^2$ corresponds to flow bounded by the positive X axis and the positive Y axis. We have solved the problem of flow into a corner forming a 90° angle. What does the flow look like, i.e., what are v_X and v_Y? What about $Z = z^{1/n}$ for integer $n > 2$? What boundary problem would this transformation solve?

5.2.2 Line Vortex

As we have noted, there is nothing special about Cartesian coordinates, and it is often convenient to use polar variables $R \cos \phi = x$, $R \sin \phi = y$. The functions Ψ and Φ lead to the velocity fields v_R and v_ϕ. A particularly useful and important flow is known as the line vortex:

$$v_\phi = \frac{\Gamma}{2\pi R}, \quad v_R = 0. \tag{362}$$

We shall leave it to the reader to show that the potential and stream functions are

$$\Phi = \frac{\Gamma\phi}{2\pi}, \quad \Psi = -\frac{\Gamma}{2\pi}\ln(R) \qquad (363)$$

and that they satisfy the Laplace equation in polar coordinates. Note that both these functions have singularities associated with the origin, where neither ϕ nor $\ln(R)$ is well-defined. The complex potential is

$$w(z) = \Phi + i\Psi = -i\frac{\Gamma}{2\pi}\left[\ln(R) + i\phi\right] = -i\frac{\Gamma}{2\pi}\ln(z), \qquad (364)$$

a result that also follows directly from the last exercise, with $v_R = 0$. (Show this!) There is nothing special about the point we choose to call the origin, so a line vortex at $z = z_0$ is simply

$$w(z) = -i\frac{\Gamma}{2\pi}\ln(z - z_0) \qquad (365)$$

The streamlines of the vortex are circles about the point $z = z_0$.

5.2.3 Cylindrical Flow

Consider an analytic function $f(z)$, except for possible isolated singularities. Assume that all such singularities lie outside the circle $|z| = a$. The Milne-Thompson theorem states that the function

$$w(z) = f(z) + \overline{f\left(\frac{a^2}{\bar{z}}\right)} \qquad (366)$$

has the same singularites as $f(z)$ outside the curve $|z| = a$, and that the circle $|z| = a$ is a streamline. (Here the bar denotes complex conjugate.) To see this, note first that all singularites of $f(z)$ lie outside $|z| = a$, and thus inside of this circle for the function $f(a^2/\bar{z})$. (The modulus of the argument of f must be greater than a for there to be a singularity.) On the circle $z\bar{z} = a^2$, w becomes

$$w(z) = f(z) + \overline{f(z)}, \qquad (367)$$

whose imaginary part (namely, the stream function) is a constant (namely, zero). Hence, the circle $|z| = a$ is itself a streamline.

We will not need the Milne-Thompson theorem in its full generality, but will instead work with the simple function

$$w(z) = U\left(z + \frac{a^2}{z}\right), \qquad (368)$$

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which corresponds to uniform flow at large $|z|$ and has the circle $|z| = a$ as a streamline. In other words, it is the solution to the problem of potential flow around a cylinder! (The region $|z| < a$ is of no interest.) If we want the flow to approach the cylinder by an angle α relative to the x-axis, we apply the circle theorem to $w = Uze^{-i\alpha}$, and obtain

$$w(z) = U \left(ze^{-i\alpha} + \frac{a^2 e^{i\alpha}}{z} \right) \tag{369}$$

In polar coordinates $z = Re^{i\phi}$, so the $\alpha = 0$ flow is

$$\Phi = U \left(R + \frac{a^2}{R} \right) \cos \phi, \quad \Psi = U \left(R - \frac{a^2}{R} \right) \sin \phi, \tag{370}$$

corresponding to the real and imaginary parts of w. Thus,

$$v_R = U \left(1 - \frac{a^2}{R^2} \right) \cos \phi, \quad v_\phi = -U \left(1 + \frac{a^2}{R^2} \right) \sin \phi. \tag{371}$$

On the surface of the cylinder $R = a$,

$$v_R = 0, \quad v_\phi = -2U \sin \phi \tag{372}$$

so that there is "slip" past the surface. In a real cylinder, viscous effects would ensure that both v_ϕ and v_R vanish at the surface. Note that although there is slip, there is no circulation: the line integral of v_ϕ round the cylinder is zero.

But our solution is not unique! We may superpose a line vortex (364) without violating the boundary conditions either at the cylinder's surface or at infinity. In this way, we can generate solutions with any circulation around the cylinder. Since the sum of two analytic functions is obviously analytic (in this sense our technique is linear), we simply add the vortex solution to v_ϕ:

$$v_\phi = -U \left(1 + \frac{a^2}{R^2} \right) \sin \phi + \frac{\Gamma}{2\pi R}. \tag{373}$$

This has circulation Γ. How is it that we have TWO solutions to the Laplace equation with the same boundary conditions? It is because the potential Φ is double-valued for the vortex. The uniqueness theorem does not hold if Φ is singular—even if the derivatives of Φ lead to perfectly physical velocity fields!

5.3 Force Exerted by a Flow

Let us calculate the pressure P on a cylinder for the more general solution in the previous section with non-zero circulation. The Bernoulli constant is

$$P + \frac{1}{2}\rho v_\phi^2 = \text{constant} \tag{374}$$

on the surface of the cylinder. Thus,

$$\frac{P}{\rho} = \text{constant} - 2U^2 \sin^2 \phi + \frac{U\Gamma}{\pi a} \sin \phi \tag{375}$$

The radially inward force per unit length along the cylinder is $Pa\,d\phi$. The x and y components are obtained by multiplying this force respectively by $-\cos\phi$ and $-\sin\phi$. Upon integrating over ϕ from 0 to 2π, the x force component is seen to vanish. This is physically obvious since the pressure on the surface depends only upon y. Moreover, *only the circulation term contributes to the y force.* The average of $\sin^2 \phi$ is $1/2$, and the net y force is

$$F_y = -\rho U \Gamma \tag{376}$$

Therefore, there is a positive lift force if there is *negative* (clockwise) circulation around the cylinder ($U > 0$). This relationship between circulation and lift is the fundamental reason that airplanes fly. Airplane wings are not cylinders, however, and we will shortly have a better understanding of why they have the shapes they do.

On one hand, the vanishing of the x force is mathematically obvious because of the symmetry; on the other hand, it seems to contradict all common sense and experience. Don't we feel a retarding force when the wind blows?? The next time you are struggling to walk in a strong wind, just tell yourself "There is no force, there is no force..."

In fact, there is a real force, we surely feel it, and real cylinders do as well. The point is that the solution we have just found is never realized in nature. Even though the air is not "very" viscous, there is some finite viscosity η and any finite viscosity changes the flow very near the surface of any object in a wind. Near the object, the flow changes not just by a little, but by a lot, whatever the value of η. In real fluids, there can be no tangential velocity at the surface. Instead, a *viscous boundary layer* forms. The boundary layer is present for any finite η, only the *thickness* of the layer changes with the magnitude of the viscosity. Inside the boundary layer, the velocity is much smaller than the inviscid solution above, and the flow comes to rest at the surface. Outside the boundary layer, our inviscid solution is an excellent approximation.

Does this mean that the drag force is essentially viscous? Our description above points in that direction, and for very viscous flows the drag force on a cylinder is indeed proportional to η. But for a large *Reynolds number* $\mathcal{R}$, defined as

$$\mathcal{R} = \frac{\rho U a}{\eta} = \frac{U a}{\nu} \tag{377}$$

the drag force per unit length for a cylinder is of order $\rho U^2 \times a$. (Here, $\nu = \eta/\rho$ is the so-called "kinematic viscosity.") The precise value of the viscosity only serves to determine the proportionality constant, but it is generally of order unity. The reason for this is fascinating, and introduces one of the most important and surprising concepts in fluid dynamics: the separation of a boundary layer.

What happens is that the pressure is large at the front surface and the back (or "trailing") surface, but small on the top and bottom of the cylinder. This gives rise to a sharply steepening pressure gradient as the flow in the boundary layer circulates around the cylinder, and moves along the trailing surface. A sufficiently high adverse pressure gradient makes it impossible for the boundary layer to remain fixed to the trailing surface. Instead, it is unstable to detachment. No such difficulties are associated with the front surface, where the pressure gradient is in the direction of the flow.

The detached boundary layer cannot exist happily in the middle of the fluid! Instead, it generates local turbulence and extended regions of high vorticity. All of this kinetic activity lowers the pressure near the trailing surface, relieving the adverse pressure gradient that a fluid would otherwise sense as it flows over the top of the cylinder. There is then a turbulent, low pressure wake behind the cylinder. If we examine flows of smaller and smaller viscosity (e.g., larger and larger Reynolds number) we do *not* approach the inviscid solution; rather we approach a solution in which the turbulent wake becomes more narrow, but just as vigorous, as the viscosity decreases. The reduced pressure behind the cylinder combined with the high pressure ahead of the cylinder result in a large effective drag force.

In fact, the vanishing of all drag forces when an object moves through an inviscid fluid is quite independent of the shape of the object. We will now show that in two-dimensional flow, if a body has a boundary described by a contour C, then the complex velocity potential $w(z)$ produces x and y forces F_x and F_y given by[3]

$$F_x - i F_y = \frac{i\rho}{2} \int_C \left(\frac{dw}{dz} \right)^2 \, dz. \tag{378}$$

This is known as *Blasius's Theorem.*

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[3]In two dimensions, when we speak of a "force," this is in reality a force per unit length.
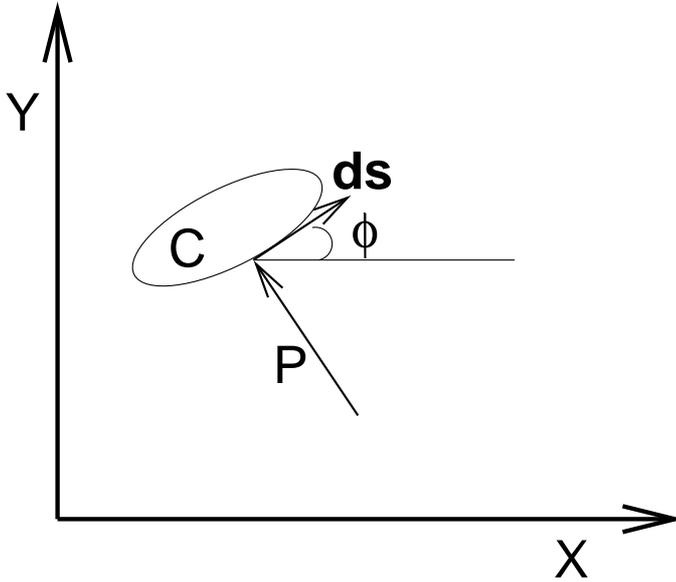
Figure 10: Geometry for Blasius's Theorem. The ellipse represents the object in the flow with boundary curve C.

To prove this elegant result, refer to figure (10). On a small boundary segment ds, the force components due to the pressure P are easily resolved:

$$dF_x - idF_y = P(-\sin\phi - i\cos\phi)\,ds \qquad (379)$$

where ϕ is the angle between the x axis and the segment ds tangent to the surface. Hence,

$$dF_x - idF_y = -iPe^{-i\phi}\,ds \qquad (380)$$

Now C is a flow streamline, so

$$\frac{dw}{dz} = v_x - iv_y = v\,e^{-i\phi} \qquad (381)$$

on C. (v is $\sqrt{v_x^2 + v_y^2}$.) The Bernoulli constant K is

$$K = P + \frac{1}{2}\rho v^2 \qquad (382)$$

so that

$$dF_x - idF_y = i(\rho v^2/2 - K)e^{-i\phi}\,ds \qquad (383)$$

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Now the K term will vanish when we do the integration: it is simply the complex conjugate of the integral of $dz = e^{i\phi}ds$ around a closed loop. Since $v^2 = (dw/dz)^2 \exp(2i\phi)$, integration over C immediately gives

$$F_x - iF_y = \frac{i\rho}{2} \int_C \left(\frac{dw}{dz}\right)^2 dz \qquad (384)$$

which is the desired result.

To apply this powerful theorem, consider first the integral of dw/dz around the surface:

$$\int_C \frac{dw}{dz}\,dz.$$

If dw/dz is free of singularities outside the object, then we may take the integration path along any surface enclosing the body. In particular, for an isolated body in a flow that is asymptotically uniform at large $|z|$, the complex velocity field is uniquely described by a Laurent series:

$$\frac{dw}{dz} = U + \frac{a_1}{z} + \frac{a_2}{z^2} + ... \qquad (385)$$

a_1 can be related to the circulation Γ as follows. Begin with

$$\frac{dw}{dz}dz = (v_x - iv_y)(dx + idy) = (v_x dx + v_y dy) + i\left(\frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial y}dy\right) \quad (386)$$

integrated around the boundary. In the imaginary part, we have replaced the velocity components by their stream function representation. But the stream function Ψ is a constant on the boundary, and thus the imaginary part of the above vanishes identically. Only the real part remains. Its integral is the circulation around the object, Γ.

Next, evaluate the same integral of dw/dz, and use the residue theorem. The result must be $2\pi i a_1$. We have therefore shown that

$$\Gamma = 2\pi i a_1. \qquad (387)$$

Finally, to evaluate the forces, we need to evaluate the closed path integral over $(dw/dz)^2$. Only one term survives after squaring (385) and integrating, namely the integral over $2Ua_1/z$. Hence

$$\int_C \left(\frac{dw}{dz}\right)^2 dz = \int_C \frac{2Ua_1}{z}\,dz = 4\pi i U a_1 = 2U\Gamma \qquad (388)$$

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Blasius's theorem then gives immediately

$$F_x = 0, \quad F_y = -\rho U \Gamma \tag{389}$$

This is what we found explicitly for the cylinder, now seen to be true much more generally.

The fact that inviscid motion through a fluid gives rise to no drag forces was viewed as paradoxical in the early days of fluid mechanics, and it is still referred to in just that way: d'Alembert's paradox. It is really a theorem, not a paradox! In fact it holds in three dimensions as well, because the momentum flux of streamlines is the same at large upstream and downstream distances. Drag forces would simply not exist for inviscid fluids. In fact, with well-designed streamlining (no detached boundary layers!), drag forces can be made to be very small, even in real flows.

The lift force is given by $F_y = -\rho U \Gamma$, as result that is known as the *Kutta-Joukowski lift theorem.* It holds for two dimensional flow past an object of any shape. Its most important application is to the theory of flight. Negative circulation around an airplane wing (i.e., in the clockwise sense) produces positive lift ($U > 0$). The question now becomes, what is the circulation Γ? Although for a cylinder we were free to choose Γ, it will turn out that for less symmetric, more winglike shapes, only one value of Γ allows the flow to be free of singularities! This is the value that airplanes choose. (Airplanes are not stupid.)

5.4   Conformal Mapping and Flight

Consider the so-called *Joukowski transformation*:

$$Z = z + \frac{c^2}{z} \tag{390}$$

where c^2 is a real positive constant. Do not confuse this with with equation (368), which is associated with the velocity field around a cylinder. Here we are using the same mathematical function to describe a *coordinate transformation.* The inverse transform is

$$z = \frac{Z}{2} + \sqrt{\frac{Z^2}{4} - c^2} \tag{391}$$

We have chosen the + sign so that $z = Z$ at large $|z|$, and we must insert a branch cut in the Z plane between $Z = 2c$ and $Z = -2c$ to avoid the square root singularity. Don't worry: the branch cut will generally be inside the body.

Consider the effect of this transformation on the circle $z = ae^{i\phi}$, the surface streamline for a cylinder. In the $Z = X + iY$ plane,

$$X + iY = ae^{i\phi} + \frac{c^2}{a}e^{-i\phi} = \left(a + \frac{c^2}{a}\right)\cos\phi + i\left(a - \frac{c^2}{a}\right)\sin\phi. \qquad (392)$$

Therefore, the circle $z = ae^{i\phi}$ maps into

$$\frac{X^2}{(a + c^2/a)^2} + \frac{Y^2}{(a - c^2/a)^2} = 1. \qquad (393)$$

This is an ellipse in the Z plane, with semi-major axis $a + c^2/a$, and semi-minor axis $a - c^2/a$. Thus, this transformation enables us to find the flow around an elliptical cylinder, once it is known for a circular cylinder.

Consider the case in which uniform flow approaches from an angle α. The solution for a circular cylinder has been worked out in in equation (369). Moreover, adding a line vortex to the problem does not change a cylindrical streamline surface. With an imposed vortex of circulation Γ, the elliptical solution may be written in parameterized form:

$$w = U\left(ze^{-i\alpha} + a^2e^{i\alpha}/z\right) - i\Gamma(\ln z)/(2\pi), \quad z = \frac{Z}{2} + \sqrt{\frac{Z^2}{4} - c^2} \qquad (394)$$

(Note that rotation by α in the line vortex logarithm just gives an unimportant additive constant to w.) From this solution, we may of course derive the general velocities v_X and v_Y, but it is more interesting to go immediately to the case $c \to a$. Then our ellipse collapses to become a plate along the X-axis of length $L = 4a$. An easy way to study the Z plane velocity fields is

$$\frac{dW}{dZ} = \frac{dw/dz}{dZ/dz} = \frac{U\left(e^{-i\alpha} - a^2e^{i\alpha}/z^2\right) - i\Gamma/(2\pi z)}{1 - a^2/z^2} \qquad (395)$$

Something new has entered into the problem: the velocity field is in general singular at the ends of the plate, corresponding to $z = \pm a$, or $Z = \pm 2a$. But by choosing the circulation properly, both the numerator and the denominator can be made to vanish simultaneously. For negative Γ, corresponding to lift, the trailing edge $Z = 2a$ $(z = a)$ can be made smooth and free of singular behavior, if

$$\Gamma = -4\pi Ua\sin\alpha. \qquad (396)$$

The leading edge is *still* singular. Can we find a flow that retains an upward lift force and is free of singularities at both edges?

Yes. The trick is beautiful but rather technical, so we refer the interested reader to Acheson for details. The basic idea is that the leading edge singularity can be removed by reshaping the surface from that of a plate, with $\pm Z$ symmetry, to something that more resembles a tear drop: blunt and rounded at the leading edge, smooth and tapering to a point at the trailing edge. This is, in fact, the shape of aircraft wings. For this nonsymmetrical shape, the leading edge singularity is actually *inside* the wing, while the trailing edge singularity is eliminated when the flow around the wing achieves the above value for the circulation:

$$\Gamma = -\pi U L \sin \alpha, \tag{397}$$

where $L = 4a$ is the full width of the wing. The Kutta-Joukowski lift theorem gives

$$F_Y = \pi \rho U^2 L \sin \alpha \tag{398}$$

the classical equation relating the upward force to the flow velocity and angle α of attack.

In practice, equation (398) works well for small α. Once the angle of approach becomes steeper than a few degrees, the boundary layer separates, the pressure behind the wing drops, and there is a great increase in the drag force. The plane stalls. This is not good. Needless to say, pilots are very careful to maintain a small angle of ascent/descent when taking off and landing!

A classical problem with flight is ice forming on the wings. Besides adding weight, which is of secondary importance, a layer of ice changes the aerodynamics, causing the boundary layer to separate earlier, reducing lift and increasing drag. Many plane crashes have occurred this way, and reputable airlines are extremely careful to de-ice their aircraft before take-off. In flight, at altitudes above $18,000$ feet (5500 m) the air is extremely dry and icing is not a problem. Modern jet aircraft generally fly at altitudes well above this, except when hovering at low altitudes waiting to land at a crowded airport. As recently as 1994, a plane crashed in the US while waiting to land in Chicago, because of ice forming on its wings.

Why does ice cause the boundary layer to separate earlier? Interestingly, it is because turbulence *within* a boundary layer can actually make the layer hold closer to the surface in the presence of an adverse pressure gradient. The increased mixing of momentum in the turbulent boundary layer acts somewhat like a large viscosity, delaying the separation of this layer. By way of contrast, the turbulence in the wake of a flow that occurs *after* a laminar boundary layer has prematurely separated is not helpful. This is bad turbulence.

You may notice that airplane wings sometimes have small, raised metal shapes, whose role is to induce boundary layer turbulence, which in this case is "good." (The shapes are called vorticity generators; a similar role is played

by the dimples [petits trous] on a golf ball.) Icing of the wings makes it more difficult for this good turbulence to form, so the boundary layer separates too soon, and the ensuing bad turbulence in the wake of the air flow increases the drag.

6 Vortex Motion

Consider the circulation integral

$$\Gamma = \int_{C(t)} \boldsymbol{v} \cdot \boldsymbol{dx} \tag{399}$$

where $C(t)$ is a closed circuit consisting of fluid particles moving with the flow. The path is described by $\boldsymbol{x}(x_0, t)$, where x_0 labels the fluid element at time $t = 0$. Then

$$\Gamma = \int \boldsymbol{v} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial x_0} \right)_t dx_0, \tag{400}$$

since of course the integration is done at fixed time. If we take the time derivative following the fluid elements $(\partial/\partial t)_{x_0}$, there obtains

$$\frac{d\Gamma}{dt} = \int \left(\frac{\partial \boldsymbol{v}}{\partial t} \right)_{x_0} \cdot \left(\frac{\partial \boldsymbol{x}}{\partial x_0} \right)_t dx_0 + \int \boldsymbol{v} \cdot \left(\frac{\partial}{\partial x_0} \right)_t \left(\frac{\partial \boldsymbol{x}}{\partial t} \right)_{x_0} dx_0 \tag{401}$$

But $(\partial \boldsymbol{x}/\partial t)_{x_0}$ is just the velocity of a fluid element, $\boldsymbol{v}$! The final integral is therefore over a perfect derivative of v^2, and since we begin and end at the same point in the flow, it vanishes. As for the first integral, the time derivative of the velocity follows a fluid element, so it is precisely the Lagrangian derivative D/Dt that appears in the equation of motion. If the quantity dP/ρ is expressible as dH (i.e. an enthalpy function exists), and all external forces are derived from a potential function χ, then

$$\frac{d\Gamma}{dt} = -\int_{C(t)} \boldsymbol{\nabla}(H + \chi) \cdot \boldsymbol{dx} = 0, \tag{402}$$

since the integrand is an exact gradient and the circuit begins and ends at the same point. This is Kelvin's circulation theorem, the most important result in vorticity theory. Notice that the circuit must move with the fluid: the theorem is false if we hold the circuit fixed in space. Note also that the theorem holds for a viscous fluid, provided that we take our one-dimensional path entirely in a region where viscosity is negligible. An area bounded by our circuit could, in principle, contain regions of very high viscosity. As long
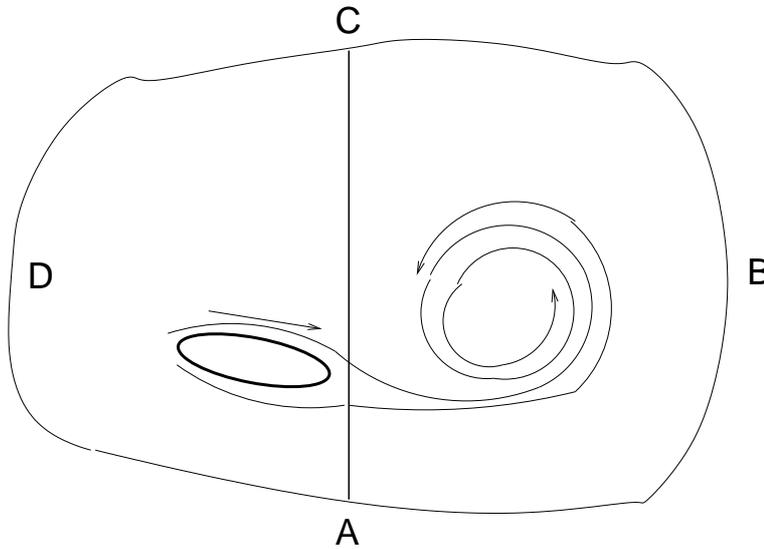
Figure 11: Aircraft wing is shown on the left as thick lined ellipse. Region of vorticity is restricted to flow near the wing and in the trailing vortex. In these regions, the circulation is equal and opposite.

as our path is inviscid, the calculation is valid. Finally, we note that the area bounded by the circuit need not be simply connected. In these respects, this proof differs from the one we presented in section 3.4, which made use of Stoke's theorem and required inviscid flow throughout the area bounded by the curve.

This theorem has interesting implications for the take-off of an aircraft. We know from our work in the last chapter that circulation is needed around the wing to generate lift. Imagine a large closed circuit well away from the wing at all times. Around this circuit the circulation must always remain zero. But near the wing we know that there is circulation. If we divide our path into two circuits shown in figure (11) then any circulation calculated around circuit ABCA must be exactly equal and opposite to the circulation calculated around circuit ACDA. In fact, aircraft generate a trailing vortex which is shed behind the wing in the process of generating vorticity of the opposite sign around the wing. This trailing vortex is crucial to generate lift! The local generation of vorticity implies a central role for viscosity, no matter how small the latter may be. Vorticity is continuously shed during takeoff, and the circulation around the wing is built up. When the circulation takes on the critical value $-\pi U L \sin \alpha$, time-steady, nonsingular flow is possible, and vortex shedding stops.

The next time you fly in a plane, you can explain all this to the person

sitting next to you...

6.1 Vorticity is Local

As we discussed in section 3.4 (and implied by Kelvin's theorem), vorticity field lines are frozen into the fluid. This has the consequence that a tube whose surface is defined by such lines remains well-defined as the flow evolves: fluid elements on a vortex line stay on the same vortex line. Remember as well that fluid may rotate around a central point, have a finite circulation integral, but *locally* have no vorticity! In fact, regions of locally intense circulation tend to have all their vorticity confined to a small core.

Atmospheric vorticity is highly localized, and can be devastating. Tornados are generated as trailing vortices from much larger regions of low pressure ('cyclones'). Tornados are first seen as vortex tubes, rather like a funnel, extending from large thunderclouds. Relative shear between the cloud and the vortex tube shrinks the tube's cross section as the length is extended. Vorticity is conserved, and there is a corresponding rise in the circulation around the tube.

James Stone, a well-known numerical astrophysicist, once described to me an episode with highly localized atmospheric vorticity that involved him personally! When he was at the University of Maryland near Washington DC, a summer storm suddenly became extremely intense while he was driving home. He deemed it prudent to stop his car and take what shelter he could by the side of the road. With his face pressed against the ground he felt a huge roar pass over him, and when he looked up, cars *on the other side of the road* had been turned over and hurled violently. Though just adjacent to the disturbance, he and his side of the road escaped serious damage. A tornado had passed within 10 meters of him.

6.2 Motion of Isolated Vortices

In the old days, the phenomenon of vortex motion could have been demonstrated as follows. I would come to class with a nice, expensive cigar (Cuban please), light it, and then draw in smoothly. I would next let out a sharp, small burst of air from the back of my throat. The emerging smoke would form a viscous boundary layer, closely following the edge of my mouth, then curl back on itself as it detached and moved into the surrounding inviscid air. A region of persistent vorticity is generated in the inviscid medium, and a smoke ring is born.

But I can't do that any more. Not only is it politically incorrect to

smoke in class, in the twenty-first century all educational experience is via the internet. So go to Google and type in "smoke rings". Or just visit

www.woodrow.org/teachers/esi/1999/princeton/projects/fluid_dynamics/vortex.html

Vortex rings are remarkably coherent: they can interact with one another, yet still retain their individual identities.[4] They are common in nature, sometimes even forming above volcanoes. An analysis of their motion which involves the interaction with an inviscid background still presents a technical challenge, but fortunately the problem of a *spherical* vortex is easily analyzed. It is harder to generate in the lab than a ring vortex, but its theoretical properties are nicely illustrative.

6.2.1   Irrotational Flow Around a Sphere

We begin with the irrotational flow surrounding our sphere of vorticity. At large distances, the flow is uniform along the z axis at velocity U. In terms of spherical radius r and spherical angle (relative to the z axis) θ,

$$v_r = U \cos \theta, \qquad v_\theta = -U \sin \theta \tag{403}$$

This motivates trying a solution of the form

$$\boldsymbol{v} = U \left[A(r) \cos \theta \boldsymbol{e_r} - B(r) \sin \theta \boldsymbol{e_\theta} \right] \tag{404}$$

Then,

$$0 = \boldsymbol{\nabla \cdot v} = \frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial (v_\theta \sin \theta)}{\partial \theta} = \cos \theta \left[\frac{1}{r^2} \frac{d}{dr}(r^2 A) - \frac{2B}{r} \right] \tag{405}$$

Hence,

$$v_\theta = -B(r) U \sin \theta = -\frac{U \sin \theta}{2r} \frac{d(r^2 A)}{dr} \tag{406}$$

The flow is irrotational, so $\boldsymbol{\omega} \equiv \boldsymbol{\nabla \times v} = 0$. The curl has only a ϕ component, however. Therefore,

$$\omega_\phi = \frac{1}{r} \left[\frac{\partial (r v_\theta)}{\partial r} - \frac{\partial v_r}{\partial \theta} \right] = 0. \tag{407}$$

---

[4]In this respect, vortex rings are similar to the solitary waves we encountered in chapter 4. Interestingly, both vortices and solitary waves (solitons) have at one time or another been used as models for elementary particles.

Substituting our expressions for v_r, v_θ, we find

$$0 = -\frac{1}{2}\frac{d^2(r^2 A)}{dr^2} + A \qquad (408)$$

This is easily solved:

$$A(r) = C_1 + \frac{C_2}{r^3} \qquad (409)$$

where C_1 and C_2 are integration constants. Since $A \to 1$ as $r \to \infty$, $C_1 = 1$. We also require $v_r = 0$ at the spherical surface, $r = a$. (There is no such restriction on v_θ for an inviscid flow.) Hence $C_2 = -a^3$. Our solution is therefore

$$v_r = U \cos\theta \left(1 - \frac{a^3}{r^3}\right), \qquad v_\theta = -U \sin\theta \left(1 + \frac{a^3}{2r^3}\right) \qquad (410)$$

Note the slip velocity $v_\theta = -(3/2)U \sin\theta$ at the surface of the sphere. (Compare v_r and v_θ with their counterparts [371] for cylindrical flow.)

6.2.2 Matching Spherical Vortex

We seek an interior solution with vorticity that matches smoothly on to the exterior flow. Under steady conditions, the vorticity equation is

$$\boldsymbol{\nabla} \times (\boldsymbol{\omega} \times \boldsymbol{v}) = 0. \qquad (411)$$

This is equivalent to

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{v}. \qquad (412)$$

This is nicely descriptive of frozen-in vorticity: the left side is the rate of change of a fluid element, the right side is the rate at which a vortex line is sheared by velocity gradients along its length.

Moreover, if the flow is independent of ϕ, only the ϕ component ω of the vorticity $\boldsymbol{\omega}$ is present, and this satisfies

$$(\boldsymbol{v} \cdot \boldsymbol{\nabla})\left(\frac{\omega}{r \sin\theta}\right) = 0 \qquad (413)$$

That is, $\omega/(r \sin\theta)$ is a constant for each element of fluid. (It is actually easiest to prove this using equation (412) in cylindrical coordinates R and z, since ϕ in both systems is identical. At the end of the calculation, switching to spherical coordinates just means identifying $R = r \sin\theta$.)

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The simplest flow that one can imagine is when *all* fluid elements have the same constant for $\omega / r \sin\theta$. Then $\omega = Cr \sin\theta$, with C to be determined.[5] Since we wish to match the flow at the spherical surface, we look for solutions with the same angular dependence on $\cos\theta$ and $\sin\theta$ as our exterior solution. Moreover, the flow is divergence-free in the sphere, so that equations (404) and (406) still hold:

$$v_r = U A_i(R) \cos\theta, \qquad v_\theta = -\frac{U \sin\theta}{2r} \frac{d(r^2 A_i)}{dr} \qquad (414)$$

where A_i is the interior counterpart to A. Then the ϕ-component of $\boldsymbol{\omega}$ is,

$$\frac{1}{r} \frac{\partial(r v_\theta)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} = \frac{U}{r} \left[-\frac{1}{2} \frac{d^2(r^2 A_i)}{dr^2} \sin\theta + \sin\theta A_i \right] = \omega = Cr \sin\theta \quad (415)$$

The $\sin\theta$ factor cancels out nicely. The resulting differential equation is

$$\frac{Cr^2}{U} = -\frac{1}{2} \frac{d^2(r^2 A_i)}{dr^2} + A_i \qquad (416)$$

The solution is

$$A_i = C_1 + \frac{C_2}{r^3} + C_p r^2 \qquad (417)$$

where C_1 and C_2 represent integration constants of the homogeneous part of the differential equation, and C_p is the coefficient of the particular solution. (Since C is as yet undetermined, we are free to chose C_p as we like.)

Now $C_2 = 0$, because the solution cannot be singular at the origin. Furthermore, we must have v_r vanish at the surface, which constrains C_p. Our solution must take the form

$$A_i = C_1 \left(1 - \frac{r^2}{a^2} \right). \qquad (418)$$

The velocity v_θ is then calculated to be:

$$v_\theta = -C_1 U \left(1 - \frac{2r^2}{a^2} \right) \sin\theta \qquad (419)$$

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[5]Don't be mislead from the form of ω into thinking that the vorticity has the "same sign" everywhere. A ϕ vector of course points in different directions, depending upon the value of ϕ!
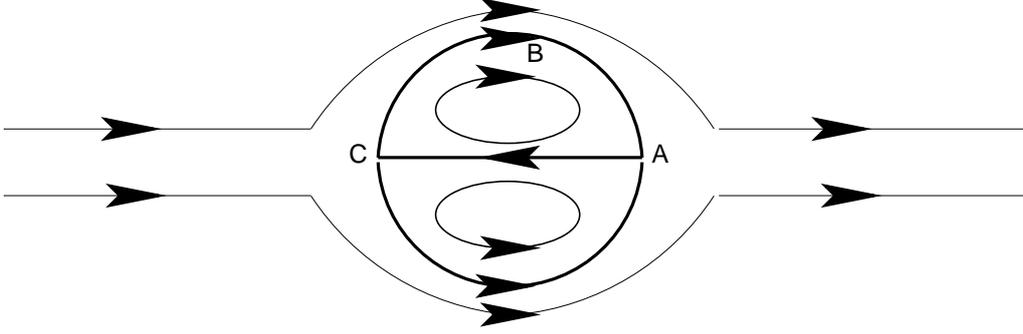
Figure 12: Schematic diagram of flow around a spherical vortex. In the rest frame of the vortex, flow enters from left and exits at right with velocity U. The maximum vorticity path within the sphere is ABCA, with circulation $\Gamma = -5Ua$, where a is the sphere radius.

At $r = a$, this must equal the slip velocity $-3U \sin \theta / 2$ that we found above, so $C_1 = -3/2$, and

$$v_r = -\frac{3U}{2} \left(1 - \frac{r^2}{a^2} \right) \cos \theta \qquad v_\theta = \frac{3U}{2} \left(1 - 2\frac{r^2}{a^2} \right) \sin \theta \qquad (420)$$

The vorticity in the sphere is calculated to be

$$\omega = -\frac{15Ur}{2a^2} \sin \theta \qquad (421)$$

The streamlines are shown in figure (12). The curve of maximum circulation goes over a great circle over a hemisphere of constant ϕ for θ going from 0 to π, then through the diameter of the sphere. Using Stokes's theorem to convert this into an area integral of ω gives

$$\Gamma_{max} = -\frac{15U}{2a^2} \int_0^\pi \sin \theta \, d\theta \int_0^a r^2 \, dr = -5Ua. \qquad (422)$$

The interior of the sphere is a collection of elliptical or hemispherical vortices, embedded in meridional planes. (This is a plane inside a sphere with $\phi =$ constant.) In a frame in which the ambient irrotational fluid is at rest, this spherical vortex region propels itself through the fluid at a speed $\Gamma_{max}/5a$! If an obstacle or secondary flow impedes the motion of the spherical vortex, it will tend to conserve its circulation, and increase its radius. Hence, isolated vortices expand when they slow down and contract when they accelerate.

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6.2.3 Inertial Drag of a Sphere by an Ideal Fluid

This section contains optional advanced material, included for your interest.

We take a small digression to discuss a very interesting application of our inviscid external spherical flow solution. In our discussion of d'Alembert's paradox, the flow was time-steady. But what if we had an external force on the sphere that caused it to oscillate? Would there *still* be no drag force?

In fact there is such a force, caused by interaction between the fluid and the moving body. Unlike the viscous drag force we shall encounter later, this force is entirely nondissipative and proportional to the acceleration, not the velocity! It thus acts to enhance the "effective mass" of the body. This idea, that an effective mass can arise out of interactions with an extended field, is a seminal concept of modern elementary particle physics.

Let us begin with a tangible problem: a sphere in an ideal fluid. The velocity fields (410) are derivable from a potential function:

$$\boldsymbol{v} = \boldsymbol{\nabla}\Phi, \quad \Phi = U\cos\theta\left(r + \frac{a^3}{2r^2}\right) \tag{423}$$

The leading order behavior of Φ in a frame in which the fluid is at rest is

$$\Phi(\text{rest}) = U\cos\theta\frac{a^3}{2r^2} = -u\cos\theta\frac{a^3}{2r^2} \tag{424}$$

where we have replaced U, the fluid velocity in the frame in which the sphere is at rest, with $-u$, where u is the velocity of the sphere in the fluid rest frame. Note that the time dependence of r caused by the moving sphere is negligible as $r \to \infty$. In the remainder of this section, $\boldsymbol{v}$ will represent the fluid velocity in the frame in which the fluid is at rest at infinity, $\boldsymbol{u}$ will be the velocity of the sphere, Φ_∞ will mean $\Phi(\text{rest})$ above, and Φ will be the exact velocity potential.

It is possible to calculate the total energy of the fluid with explicit knowledge of its velocity only at ∞. The kinetic energy of the fluid is

$$E = \frac{\rho}{2}\int v^2\,dV = \frac{\rho}{2}\int u^2\,dV + \frac{\rho}{2}\int (\boldsymbol{v}+\boldsymbol{u})\cdot(\boldsymbol{v}-\boldsymbol{u})\,dV \tag{425}$$

where the integration is over a large sphere of radius R (which will tend to infinity at the end of the calculation), minus the volume $V_0 = 4\pi a^3/3$ occupied by the small sphere. Now

$$(\boldsymbol{v}+\boldsymbol{u})\cdot(\boldsymbol{v}-\boldsymbol{u}) = (\boldsymbol{v}-\boldsymbol{u})\cdot\boldsymbol{\nabla}(\Phi+\boldsymbol{u}\cdot\boldsymbol{r}) = \boldsymbol{\nabla}\cdot[(\Phi+\boldsymbol{u}\cdot\boldsymbol{r})(\boldsymbol{v}-\boldsymbol{u})] \tag{426}$$

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since the divergences of $\boldsymbol{v}$ and $\boldsymbol{u}$ both vanish. Notice now that Φ here is the exact solution, not the dipole solution Φ_∞.

The volume integral of $v^2 - u^2$ is therefore

$$\int (v^2 - u^2)\, dV = \int (\Phi + \boldsymbol{u} \cdot \boldsymbol{r})(\boldsymbol{v} - \boldsymbol{u})\cdot d\boldsymbol{A} \tag{427}$$

where the surface is now over the outer sphere of radius R and the small surface of radius a. But the integral over the inner surface area vanishes because the normal component of $\boldsymbol{v}$ is exactly the normal component of $\boldsymbol{u}$ at the inner surface. (The fluid doesn't penetrate the sphere, nor does it separate and leave a vacuum!) Hence the dot product with $d\boldsymbol{A}$ vanishes.

We are left with only the surface at infinity. (The reason we did not evaluate $\int v^2$ directly is that it would *not* lead to an integral only over a surface at infinity.) With

$$\Phi_\infty = -u\cos\theta \frac{a^3}{2R^2}, \quad v_r(R) = u\cos\theta\frac{a^3}{R^3}, \tag{428}$$

and $\boldsymbol{u} \cdot \boldsymbol{r} = uR\cos\theta$, we have

$$(\Phi + \boldsymbol{u} \cdot \boldsymbol{r})(\boldsymbol{v} - \boldsymbol{u})\cdot d\boldsymbol{A} = \left(\frac{3a^3}{2} - R^3\right)u^2\cos^2\theta\, d\Omega \tag{429}$$

where we have dropped the term $\Phi_\infty v_r(R)$ (why?), and $d\Omega$ is a unit of spherical solid angle. Since the average value of $\cos^2\theta$ of the surface a sphere is $1/3$, we find

$$\int (\Phi + \boldsymbol{u} \cdot \boldsymbol{r})(\boldsymbol{v} - \boldsymbol{u})\cdot d\boldsymbol{A} = 2\pi a^3 u^2 - \frac{4\pi u^2 R^3}{3} \tag{430}$$

and hence

$$\int [u^2 + (v^2 - u^2)]\, dV = \frac{4\pi u^2}{3}(R^3 - a^3) + 2\pi a^3 u^2 - \frac{4\pi u^2 R^3}{3} = \frac{2\pi}{3}a^3 u^2 \tag{431}$$

and the kinetic energy is

$$E = \frac{\pi}{3}a^3\rho u^2 \tag{432}$$

The fluid therefore acts as though it had a mass of $2\pi\rho a^3/3$ and a momentum of $\boldsymbol{p} = 2\boldsymbol{u}\pi\rho a^3/3$! In particular, if outside force $\boldsymbol{f}$ acts on the sphere, the equation of motion is

$$\left(\frac{d\boldsymbol{p}}{dt}\right)_{sphere} + \left(\frac{d\boldsymbol{p}}{dt}\right)_{fluid} = \left(M_{sph} + \frac{2\pi\rho a^3}{3}\right)\frac{d\boldsymbol{u}}{dt} = \boldsymbol{f} \tag{433}$$

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We must add half the density of the liquid $\rho/2$ to the mass density of the sphere to get the proper equation of motion.

The drag force we have just calculated is inertial, not dissipative, and makes the sphere appear to be more massive then it really is. Of course the word "really" is somewhat ambiguous. If we had no way to remove the sphere from the water, it would not be possible to tell the difference between the intrinsic mass of the sphere and the effective mass that is the coefficient of $d\boldsymbol{u}/dt$ in the above expression. Indeed, our result formally yields a finite mass for the sphere even if the "bare mass" M_{sph} were zero! With this simple hydrodynamics example, we can get some sense of why, for example, an electron migrating through a crystal acts as though it were a considerably more massive particle. Elementary particles as well are thought to acquire their mass as a consequence of interactions with a different sort of a background medium: the vacuum itself, which in quantum field theory is no longer an inert entity. In this case, the interactions are more complex, and can either enhance or decrease the bare mass. It is thought that infinite (or at least very large) bare masses transform themselves itself into finite, measurable masses via these kinds of interactions.

6.3   Line Vortices and Flow Past a Cylinder

6.3.1   Vortex Pair

Back to vortices in the complex z plane.

In two dimensions, a vortex pair located at $z = \pm d$, with equal and opposite circulations $\pm\Gamma$ has the complex potential function

$$w(z) = -\frac{i\Gamma}{2\pi}\left[\ln(z-d) - \ln(z+d)\right] \tag{434}$$

As with a spherical vortex, a line vortex pair remains coherent but cannot be at rest relative to the background medium. Each vortex core feels the induced motion of the other, a velocity of $\Gamma/(4\pi d)$ in the $-y$ direction. Thus, the flow in which the vortices remain stationary is given by:

$$w(z) = -\frac{i\Gamma}{2\pi}\left[\frac{z}{2d} + \ln\left(\frac{z-d}{z+d}\right)\right] \tag{435}$$

In general the propagation speed of an ensemble of vortices may be calculated by choosing any one vortex, and superposing the contributions of the velocity field from all other vortices at the chosen vortex core. For this technique to work, the problem must be highly symmetric, with the calculated velocity independent of the chosen vortex.
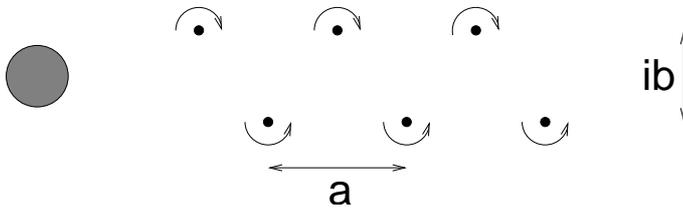
Figure 13: Model of Von Kármán vortex street. Vortex cores are spaced regularly with horizontal spacing a and vertical separation ib in the complex plane. See text for further details.

6.3.2 Flow Past a Cylinder

A classic problem of fluid dynamics is the study of flow past a cylinder. Viscous flows (the true, nonideal flows found in nature) form a boundary layer near the surface of the cylinder. At high Reynolds number, this layer detaches at the trailing end and wraps itself into two distinct regions of differing vorticity, above and below the symmetry midplane of the cylinder. If the Reynolds number Re is below ~ 30, that is the end of the story, whereas for Re beyond 2000, the vortices dissolve into a turbulent wake. In between, at values of $Re \sim 200$, the flow behaves in a remarkable way.

The vortices behind the cylinder are stretched behind the cylinder in the direction of the flow as time goes on, and this extended tail *oscillates* above and below the symmetry plane, even though the flow at infinity remains quite steady. Moreover, vortices are shed alternately from the top and bottom sides of the cylinder. They trail behind the cylinder, but NOT at the average velocity of the downstream fluid. Instead, they form a regular pattern as shown in figure (13) and chase after the cylinder, though they travel at a smaller velocity, falling farther and farther behind. We seek to calculate the velocity of this *von Kármán vortex street.*

6.3.3 A Model of the von Kármán Vortex Street.

We model the vortex street as an infinite plane of parallel line vortices, regularly spaced, with a vortex at $z = na$, $n = 0, \pm 1, \pm 2$, etc. Another plane lies parallel to the first, with vortices staggered at positions $z = (n + 1/2)a + ib$. (See figure 13).

Pick a vortex. The contributions to the local velocity from the other vortices in the same line (or plane, in three dimensions) as the chosen vortex cancel out completely. The contributions of the vortices from the other plane

also cancel out, *but only in the y direction.* In the x direction there is a reinforcement, always in the direction of negative x.

Let us choose now a vortex in the upper $x + ib$ plane. The sum of the complex potentials from all the vortices in the bottom row at a point z in the upper row is

$$w(z) = -i\frac{\Gamma}{2\pi}\left[\sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty}\ln\left(1-\frac{z}{na}\right) + \ln z\right] \tag{436}$$

We have replaced $\ln(z - na)$ with $\ln[1 - z/(na)]$, since the two forms differ only by an additive constant, which of course vanishes when we take the z derivative to obtain the velocity fields.

Now the term in square brackets is just

$$\ln\left[z\prod_{n=1}^{n=\infty}\left(1-\frac{z^2}{n^2a^2}\right)\right]. \tag{437}$$

The infinite product that appears above can be evaluated with the following elegant mathematical trick. When expanded, the product is a long polynomial, a Taylor series in z^2, as we include more and more terms. The value of the product is 1 at $z = 0$. But there can be only one Taylor series that has zeros at $z = \pm na$ and is equal to one at $z = 0$. Polynomials with exactly the same roots and multiplicity of roots must be the same function, if they have the same nonvanishing value at $z = 0$. This also holds true for a possibly infinite Taylor series generated from the factored from of the polynomial. The unique polynomial/Taylor series we seek is a familiar function:

$$\prod_{n=1}^{n=\infty}\left(1-\frac{z^2}{n^2a^2}\right) = \frac{\sin(\pi z/a)}{(\pi z/a)}. \tag{438}$$

The Taylor series for the right hand side must be the same polynomial as the expanded product on the left hand side, because they both have the same roots, the same multiplicity of roots (each root occurs once), and the same nonvanishing value at $z = 0$. Our sum of complex potentials becomes

$$w(z) = -i\frac{\Gamma}{2\pi}\ln[\sin(\pi z/a)] \tag{439}$$

where once again we omit an additive constant from the argument of the logarithm. Its z derivative is

$$\frac{dw}{dz} = v_x - iv_y = -i\frac{\Gamma}{2a}\cot(\pi z/a) \tag{440}$$

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Evaluating this at the representative vortex $z = ib + a/2$, we find that the right side is real,

$$v_x = -\frac{\Gamma}{2a}\tanh(\pi b/a), \quad v_y = 0. \qquad (441)$$

(The same result would be found at any vortex $z = ib+(n+1/2)a$, for integer n.) The vortex street moves to the left, following the cylinder, at a velocity of about $\Gamma/2a$, if a and b are comparable. In what frame is this velocity? There is no uniform motion at infinity, so this must be the frame in which the cylinder itself is moving. The vortex street therefore moves in the same direction as the cylinder, but it trails behind.

7 Viscous Flow

The most important way that real fluids differ from ideal fluids is that real fluids are viscous. Water is wet because it is viscous. Without viscosity, you would be completely dry when you emerged from a swimming pool! We have already seen the fundamental role viscosity plays in forming boundary layers and in generating vorticity, and it is now time to understand the details.

Consider a simple gas shear flow, with zero mass flux. The gas particles have no internal degrees of freedom in our approximation. The velocity is in the y direction, but depends linearly on x: $v_y = Ax$ where $A = dv_y/dx$ is constant. Since the fluid is a gas, there is a finite mean free path λ between collisions of gas particles. Thus, at any point in the flow x, there will be particles scattered from $x - \lambda$, and some from $x + \lambda$ as well. The $-$ particles will, on average, have a value of their y velocity $\sim v_y(x - \lambda)$ (read "v_y at $x - \lambda$"), since $x - \lambda$ is where the particles were most likely to have been scattered from. Similarly, the $+$ particles will have y velocity of $\sim v_y(x + \lambda)$.

Since there is no net flow of mass induced by the existence of a finite free path and a shear velocity gradient, the mass flux μ from either direction must be same. We estimate μ in each direction to be $\sim \rho c/6$, where ρ is a local mass density and c a local thermal speed. The factor of $1/6$ comes from the approximation that $1/3$ of the particles will be moving predominantly along the x direction at some instant of time, and of these only $1/2$ will be moving toward the $+$ (or $-$) direction. This gives a net y momentum flux in the x direction of

$$\frac{1}{6}\rho c[v_y(x - \lambda) - v_y(x + \lambda)] \simeq -\frac{1}{3}\rho c\lambda\frac{dv_y}{dx} \qquad (442)$$

The coefficient

$$\eta = \frac{1}{3}\rho c\lambda \qquad (443)$$

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is known as the *dynamical viscosity*, as distinguished from the *kinematic viscosity*

$$\nu \equiv \frac{\eta}{\rho} = \frac{1}{3}c\lambda \tag{444}$$

Our explicit expressions for η and ν are only approximations (the factor of $1/3$ is not very trustworthy), but they are roughly correct and in accord with laboratory data for fluids. The kinematic viscosity ν has units of (length)^2 (time)$^{-1}$, and is thus a classical diffusion coefficient.

The mean free path λ is calculated by assuming that each particle has a cross section σ for interacting with another particle. Moving through the fluid, a particle that is about to be scattered "sees" an effective area of σ presented by each target particle. The particle to be scattered is said to have moved a distance of one mean free path λ when the volume $\sigma\lambda$ contains one target:

$$n\lambda\sigma = 1, \tag{445}$$

where n is the total number density of gas particles. Therefore, $\lambda = 1/(n\sigma)$, and the viscosity η, which is proportional to $n\lambda$ does not depend on the density of the gas!

Maxwell was the first to estimate the viscosity of a gas in this manner, and was very surprised to find that η is independent of the density. This means that the drag force on a small mass is the same in a diffuse or dense gas! He checked his results experimentally. In other types of diffusion problems, however, the kinematic viscosity is more relevant than the dynamic viscosity. Kinematic viscosities for water and air are 0.01 and 0.15 cm^2 s^{-1} respectively. In this sense, air can be more viscous than water.

Notice that our estimate is sensitive to the fact that the background velocity is in a direction orthogonal to the direction in which the momentum is transported. If we were to try this argument with an x velocity propagating a momentum flux in the x direction, we would find that the momentum flux is just the pressure! The argument would go as follows:

Go into the fluid rest frame. One mean free path away, the bulk velocity would be $\pm\lambda dv_x/dx$. A particle coming from one mean free path "below" would have velocity $c_x - \lambda dv_x/dx$, where c_x is a random thermal velocity component. The momentum flux would be $(1/2)mn(c_x - \lambda dv_x/dx)^2$, the factor of $1/2$ coming from the fact that only half the particles would have random velocities with positive x. The dominant momentum flux is $(1/2)mnc_x^2$, if, as we must assume, the mean free path is small. From "above", there is a similar contribution, and the total momentum flux is nmc_x^2, which on averaging becomes the pressure, nkT. This nkT momentum flux is not associated with a direction because the problem is symmetric with respect to $+$ and $-x$.

Only a standard pressure emerges, and this has already been included in the equations of motion.

You might argue that there is a contribution at the next level of approximation, linear in λ, and that this should be the viscous stress contribution. But care must be taken. The actual distribution function of particle velocites also changes at different locations (no longer Maxwellian), and the variable c_x itself is also slightly changing, and *all* of these changes must be taken into account. Our simple and intuitive procedure is not rigorous enough to be trusted beyond the leading order term! In fact, viscous stresses proportional to to $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ *can occur*, but this requires that the molecules have internal degrees of freedom, or that there is a mixture of gases with different response times. At this level of analysis, the subject becomes very complex. For more information, see the texts by Batchelor or Landau and Lifschitz.

7.1 The Viscous Stress Tensor

How do we generalize to an unrestricted geometry from the proceeding discussion? Clearly, some care is required. The most general expression for the momentum flux would include an isotropic pressure term plus nonisotropic viscous terms proportional to velocity derivatives. But which combination of velocity derivatives should we include for the most general viscous stress?

The momentum flux takes the form of a tensor, with two indices, T_{ij}. i indicates the component of the momentum of interest, and j indicates the direction of transport of this momentum component. The pressure is isotropic, hence it must make a contribution to T_{ij} of $-P\delta_{ij}$. (The minus sign is included because it is $-\boldsymbol{\nabla}P$ that appears in the usual Euler equation.) We also expect that the viscous contribution will be proportional to a linear combination of velocity derivatives. The most general combination which acts like a tensor will be of the form

$$\sigma_{ij} \equiv \eta \left[\frac{\partial v_i}{\partial x_j} + \alpha \frac{\partial v_j}{\partial x_i} + \beta \delta_{ij} \frac{\partial v_k}{\partial x_k} \right] \tag{446}$$

where, as usual, we sum over repeated indices, and η is by definition the dynamical viscosity. (For a simple shear flow, this reduces to our earlier result.) To determine α and β, we require that the viscous stress vanish for (1) isotropic, uniform expansion; and (2) uniform rotation. Isotropic, uniform expansion ($\boldsymbol{v} \propto \boldsymbol{r}$) has no unique direction to define the orientation of a flux: the velocity field recedes homogeneously from any point chosen as the origin. If the mean flow is exactly isotropic, then the random particle velocities will also be exactly isotropic (because collisions cannot create anisotropies), and isotropic random flow cannot lead to a systematic viscous stress. The

momentum flux can be described only by an isotropic pressure term, which is already in our equations. On the other hand, a uniformly rotating flow has no shear at all. There can be no viscous stress under these circumstances.

Consider the flow $v_x = y$ and $v_y = -x$. (Any uniform rotation will just be a multiple of this flow.) Then

$$\sigma_{xy} = \eta(1 - \alpha) \tag{447}$$

Hence $\alpha = 1$. (It is easy to show that all other σ components will vanish as well.) For uniform expansion, take

$$v_x = x, \quad v_y = y, \quad v_z = z.$$

Then,

$$\sigma_{xx} = \eta(2 + 3\beta) \tag{448}$$

Hence $\beta = -2/3$. (One again, it is easily shown that all other σ components are then zero as well.) We have thus shown that

$$\sigma_{ij} = \eta \left[\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial v_k}{\partial x_k} \right] \tag{449}$$

This form of the viscous stress was first derived by Navier, and more rigorously by Stokes.

Although our discussion has focused on a dilute gas, this form of the stress holds for most ordinary liquids as well.[6] The basic reason for this is that viscous stresses tend to oppose deformation of the shape of a local volume of fluid elements. For a liquid, the divergence of $\boldsymbol{v}$ vanishes, and the viscous stress is then the most general linear superposition of velocity gradients that vanishes for uniform rotation. All other velocity profiles distort the shape of a volume moving with the flow.

The total stress tensor is defined to be

$$T_{ij} = -P\delta_{ij} + \sigma_{ij} \tag{450}$$

The equation of motion for a viscous fluid is then

$$\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \frac{\partial T_{ij}}{\partial x_j} \tag{451}$$

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[6]Manufactured liquids containing large chain molecules, like polymers or paints, can have very different viscous properties because of preferred directions. Such fluids are said to be non-Newtonian.

For an incompressible fluid, the equation of motion becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}P + \nu\nabla^2\boldsymbol{v} \tag{452}$$

This is the classical *Navier-Stokes* equation. Be careful of the $\nabla^2\boldsymbol{v}$ term in non-Cartesian coordinates! If $\boldsymbol{e_u}$ is a unit vector,

$$\nabla^2(a\boldsymbol{e_u}) = \boldsymbol{e_u}\nabla^2 a + a\nabla^2\boldsymbol{e_u} + 2\left[(\boldsymbol{\nabla}a)\cdot\boldsymbol{\nabla}\right]\boldsymbol{e_u} \tag{453}$$

Alternatively, note that if $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$,

$$\nabla^2\boldsymbol{v} = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{v}). \tag{454}$$

Finally, let us note the force df_j in the j direction on a differential surface area element with normal components dS_i due to the motion of a viscous fluid:

$$df_j = T_{ij}dS_i = -PdS_j + \sigma_{ij}dS_i = -\left(P + \frac{2\eta}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v}\right)dS_j + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)dS_i. \tag{455}$$

Exercise. Derive the explicit Navier-Stokes equation in cylindrical coordinates:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)v_R - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R} + \nu\left(\nabla^2 v_R - \frac{v_R}{R^2} - \frac{2}{R^2}\frac{\partial v_\phi}{\partial \phi}\right), \tag{456}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)v_\phi + \frac{v_R v_\phi}{R} = -\frac{1}{\rho R}\frac{\partial P}{\partial \phi} + \nu\left(\nabla^2 v_\phi - \frac{v_\phi}{R^2} + \frac{2}{R^2}\frac{\partial v_R}{\partial \phi}\right), \tag{457}$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)v_z = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \nu\nabla^2 v_z. \tag{458}$$

7.2 Poiseuille Flow

The flow of a viscous liquid through a tube was first studied experimentally by the physician Jean-Louis Poiseuille in connection with blood circulation. The problem has been one of the most important in fluid mechanics: though the flow itself is simple, its stability properties have been extremely challenging to unravel.

A tube containing a fluid with kinematic viscosity ν is subject to a fixed pressure gradient $dP/dz = P'$. The one-dimensional flow v_z depends only on radius R. We wish to calculate the velocity profile and mass flux through the tube.

As formulated, the flow automatically satisfies $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$. The boundary condition for a viscous flow is that *all* velocity components vanish at a fixed surface. This is true for any finite viscosity no matter how small, even though the boundary condition for $\eta = 0$ is that only the normal velocity component vanishes. The fact that boundary conditions change discontinuously when an infintesimal viscosity is added to the flow is what gives the study of real flows their fascinating complexity and richness. It is responsible for the formation and detachment of boundary layers and the complexity associated with the onset of turbulence. This abrupt change of boundary condition is a result that deserves a small digression.

The viscosity is an internal physical property of the fluid, whereas the boundary conditions involve the interaction between the flow and the wall. So why is it that presence of a fluid viscosity influences how a flow interacts with the wall? What happens is that on the microscopic scale of the solid surface the bumps and irregularities trap fluid in contact with the surface. But in a viscous fluid, that trapped surface layer communicates via shear stresses with the adjacent fluid layers, which then must also cease flowing. In an inviscid flow, the rest of the fluid would not care what the layer in contact with the wall is doing. In fact, in superfluids with a truly vanishing viscosity (e.g. liquid Helium), shear velocities can persist indefinitely along ordinary surfaces.

Since the $\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v}$ term vanishes, the equation of motion is

$$0 = -\frac{P'}{\rho} + \frac{\nu}{R}\frac{d}{dR}\left(R\frac{dv_z}{dR}\right), \tag{459}$$

subject to the boundary condition $v_z = 0$ at the radius of the tube $R = l$. Integrating once gives

$$\nu\frac{dv_z}{dR} = \frac{C}{R} + \frac{P'R}{2\rho}, \tag{460}$$

where C is an integration constant. Clearly $C = 0$ if the velocity is nonsingular at $R = 0$. Integrating again, and applying the no-slip boundary condition at the outer radius $R = l$ leads to

$$v_z = -\frac{P'}{4\eta}(l^2 - R^2), \tag{461}$$

where $\eta = \rho\nu$. The integrated mass flux μ is

$$\mu = \int_0^l 2\pi R v_z\, dR = -\frac{\pi P' l^4}{8\eta}, \tag{462}$$
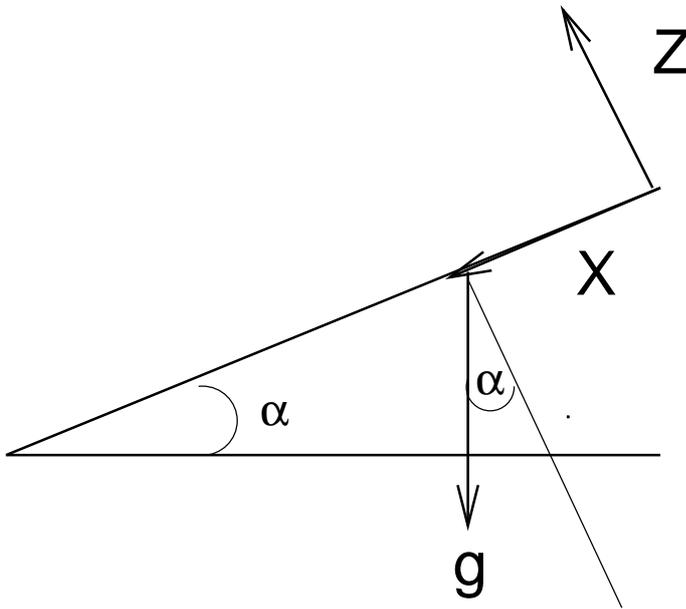
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Figure 14: Geometry for viscous flow down an inclined plane.

where the minus sign means that the velocity and the mass flux have the opposite sign of P'. The flow is proportional to the *fourth power* of the size of the opening l! (This could have been deduced on the basis of dimensional analysis.) For a healthy circulation, keep your blood viscosity low and your arteries clear.

Exercise. Consider Poiseuille flow between two concentric cylinders of inner and outer radius l_1 and l_2 respectively. Solve for the velocity profile $v_z(R)$. Show that the integrated mass flux μ is

$$\mu = \int_0^l 2\pi R v_z \, dR = -\frac{\pi P' l^4}{8\eta} \left(l_2^4 - l_1^4 \right) \left[1 - \frac{l_2^2 - l_1^2}{l_2^2 + l_1^2} \frac{1}{\ln(l_2/l_1)} \right]$$

What is formal value of the "correction factor" in square brackets if l_1 is 0.01 of l_2? 10^{-8} of l_2 (one atomic radius if l_2 is 1 cm)? Notice the exquisite sensitivity of the flow to the presence of any central cylinder of almost any finite radius.

7.3 Flow down an inclined plane

Consider the flow of a viscous fluid down an inclined plane. The upper surface is free. (The flow of hot lava down the side of a mountain comes to mind.) Let x be the Cartesian coordinate pointed downward, parallel to the slope, and z is upward, normal to the inclined surface. The inclination angle of the slope is α (see figure 14). We seek solutions of the form $\boldsymbol{v} = v_x(z)\boldsymbol{e_x}$. The pressure P is also only a function of z, since there is symmetry in x. The height of the fluid is h. The equations of motion in the x and z directions are therefore respectively

$$0 = \nu \frac{d^2 v_x}{dz^2} + g \sin \alpha \quad (x \text{ component,}) \tag{463}$$

$$0 = -\frac{1}{\rho} \frac{dP}{dz} - g \cos \alpha \quad (z \text{ component.}) \tag{464}$$

The boundary conditions are as follows. At $z = 0$, the viscous no-slip condition is $v_x = 0$. At $z = h$, the free surface, we must have pressure balance between the fluid and the atmosphere, but there is also a viscous constraint. In a viscous fluid, a velocity gradient of the form dv_x/dz would lead to momentum transport in the x direction across a $z = $ constant surface. There is no such momentum flux in the layer just above the fluid. Hence $dv_x/dz = 0$ at $z = h$. This is known as a stress-free boundary condition.

The problem is easily solved, since the equations decouple. The solution to the velocity equation with the above boundary conditions is

$$v_x = \frac{g \sin \alpha}{2\nu} z(2h - z) \tag{465}$$

and the pressure is

$$P = \rho g (h - z) \cos \alpha \tag{466}$$

taking the atmospheric pressure to be zero (to first order). The mass flux $\int \rho v_x dz$ is a quantity of some interest:

$$\int_0^h \rho v_x dz = \frac{\rho g h^3}{3\nu} \sin \alpha \tag{467}$$

Note the sensitive dependence on h.

7.4 Time-dependent diffusion

The presence of viscosity gives the equations of motion a diffusive character. It is of interest to understand how viscous stresses can impart shear motion

to a gas that is initially at rest. We seek in this section a time-dependent solution to the viscous equations of motion.

As before, we investigate one-dimensional spatial motions, with velocity $v_x(z,t)$. The x equation of motion is, in the absence of all forces but viscosity,

$$\frac{\partial v_x}{\partial t} = \nu \frac{\partial^2 v_x}{\partial z^2} \tag{468}$$

which is the classical diffusion equation. We assume that the fluid is bounded from below by a wall, which at $t = 0$ suddenly moves in the x direction at velocity U. In the frame of the wall, the adjacent fluid must be at rest, which means that in the "lab" frame this fluid moves with velocity U: the boundary condition is $v(0,t) = U$, and of course $v(\infty, t) = 0$.

To solve the partial differential equation, we search for what is known as a *self-similar* solution. This is a technique that works very smoothly when it is applicable, but in general it requires very special circumstances. In our case, we observe that the problem is so simple, there is only one parameter with dimensions that is present: ν. You might object that U is a dimensional parameter, but it just scales out of the problem. If we define a new variable $f \equiv v_x/U$, we have the same differential equation for f, and the boundary condition at $z = 0$ is $f = 1$. U is gone!

Since ν is the only dimensional parameter that really enters, there is only one *dimensionless* parameter we can construct from z, t, and ν: $\xi = z/(\nu t)^{1/2}$. We seek a solution of the form $f(\xi)$, a function that is mathematically dependent on one variable only, but that satisfies our partial differential equation in two variables. This is the trick of a self-similar solution. The name comes from the fact that the solution looks identical along curves in the zt plane for which (in this case) $t \propto z^2$. At time t, therefore, the viscous "boundary layer" near the wall has a thickness of order $(\nu t)^{1/2}$.

Self-similarity requires that only one, or at most two, dimensionless parameters be present in a problem, which is why it is not a general technique. In particular, both the equations *and the boundary conditions* must be consistent with self-similarity. When this is so, the problem is reduced from trying to find the solution of a partial differential equation to finding the solution of an ordinary differential equation, which is usually a far simpler task.

Noting

$$\frac{\partial f}{\partial t} = \frac{\partial \xi}{\partial t} f'(\xi) = -\frac{1}{2} \frac{\xi}{t} f'(\xi), \tag{469}$$

$$\frac{\partial f}{\partial z} = \frac{\xi}{z} f'(\xi), \quad \frac{\partial^2 f}{\partial z^2} = \left(\frac{\xi}{z}\right)^2 f''(\xi) \tag{470}$$

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our diffusion equation becomes

$$-\frac{1}{2}\frac{\xi}{t}f'(\xi) = \nu\left(\frac{\xi}{z}\right)^2 f''(\xi) \rightarrow f'' + \frac{\xi f'}{2} = 0. \qquad (471)$$

This is easily solved:

$$f = A + B\int_0^\xi e^{-s^2/4}\,ds \qquad (472)$$

Since $\int_0^\infty e^{-s^2/4}\,ds = \pi^{1/2}$, the boundary conditions are satisfied when

$$f(\xi) = 1 - \frac{1}{\pi^{1/2}}\int_0^\xi e^{-s^2/4}\,ds \qquad (473)$$

which completes our problem. (The integral may be expressed in terms of the special function known as the "error function". See, for example, Abramowicz and Stegun, Handbook of Mathematical Functions, page 297.)

The vorticity $\partial v_x/\partial z$ is particularly revealing,

$$\frac{\partial v_x}{\partial z} = -\frac{U}{(\pi\nu t)^{1/2}}\exp(-z^2/4\nu t) \qquad (474)$$

This is highly concentrated at $t = 0$ in manner similar to a Dirac delta function, and then spreads with time. In particular, it shows the scaling, which is true in general, that vorticity spreads over a distance L in a time of order L^2/ν.

7.5   Cylindrical Flow

In cyclindrical geometry, the Navier-Stokes equation takes on a particularly simple form for a flow of the form $\boldsymbol{v} = v_\phi(R)\boldsymbol{e_\phi}$. Taking care to include the effects of differentiation upon $\boldsymbol{e_\phi}$, the equation of motion is

$$\nabla^2(v_\phi\boldsymbol{e_\phi}) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial v_\phi}{\partial R}\right) - \frac{v_\phi}{R^2} = 0. \qquad (475)$$

This is homogeneous in R and must have power law solutions. It is a simple matter to show that

$$v_\phi = AR + \frac{B}{R} \qquad (476)$$

where A and B are integration constants. In what is known as *Couette flow*, a viscous fluid is confined between two coaxial rotating cylinders, and A

and B are determined by the no-slip condition on each cylindrical surface. The stability of Couette flow is a classical laboratory experiment in fluid mechanics. We shall speak more about it when we discuss fluid instabilities.

Note the form of our solution. $v_\phi = $ constant is definitely not a solution, whereas in Cartesian coordinates a constant velocity component *would* satisfy the viscous equation of motion. For rotational flows, we find that our solutions is a superposition of two very distinctive flows: solid body rotation, for which the vorticity $\boldsymbol{\omega}$ is constant, plus *zero-vorticity* rotation. Both of these solutions obviously satisfy

$$-\nabla^2 \boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{\omega} = 0,$$

where $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{v}$.

Exercise. What are A and B when the inner cylinder has radius R_1 and rotates at Ω_1 and the outer cylinder has radius R_2 and rotates at Ω_2?

7.6 The Stokes Problem: Viscous Flow Past a Sphere

A simple way of determining the viscosity of a fluid is to take a small solid sphere, and let it sink in a big container of the fluid in question. Measure the terminal velocity of the sphere. If the drag force on the sphere is known, the viscosity may be determined by measuring the constant terminal velocity of the sphere. The problem of determining the viscous drag force on a sphere was solved by Stokes in 1845, but mathematical subtleties plagued the subject until the 1950's! It was definitively resolved by the development of an advanced analytic technique known as asymptotic matching, which has found widespread applications in many areas of applied mathematics. Clearly this is a rich problem.

Of course, the problem is of interest for physicists as well. The classical Milikan oil drop experiment measured the charge of an electron by applying a electric field to a charged spherical small sphere of oil and noting its terminal velocity. This required a knowledge of the Stokes drag force on the drop due to the viscosity of air.

What is the drag force on a sphere moving through a viscous medium?

7.6.1 Analysis

In the low Reynolds number approximation for incompressible flow, the governing flow equations are

$$-\boldsymbol{\nabla} P + \eta \nabla^2 \boldsymbol{v} = 0 \tag{477}$$

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$$\boldsymbol{\nabla}\!\cdot\!\boldsymbol{v} = 0 \tag{478}$$

We may immediately conclude that

$$\nabla^2 P = 0, \qquad \nabla^2\boldsymbol{\omega} = 0 \tag{479}$$

where $\boldsymbol{\omega}$ is the vorticity $\boldsymbol{\nabla}\!\times\!\boldsymbol{v}$. The flow is axisymmetric and we work in a frame in which the sphere is at rest. Then at large distances from the sphere, the flow is uniform with velocity V along the z-axis. We shall work in spherical r, θ, ϕ coordinates, with origin at the center of the sphere. The flow is independent of ϕ, and $\boldsymbol{\omega} = \omega\boldsymbol{e_\phi}$ has only a ϕ component.

As $r \to \infty$,
$$\boldsymbol{v} \to V\boldsymbol{e_z} = V(\cos\theta\boldsymbol{e_r} - \sin\theta\boldsymbol{e_\theta}) \tag{480}$$

Since the flow satisfies simple linear differential equations, we expect this angular dependence to remain valid for all values of r, and this suggests trying a solution of the form

$$\boldsymbol{v} = V[A(r)\cos\theta\boldsymbol{e_r} - B(r)\sin\theta\boldsymbol{e_\theta}] \tag{481}$$

where A and B approach unity as $r \to \infty$. The flow must also vanish at the sphere; the no-slip boundary condition requries that both A and B vanish at the spherical radius $r = a$.

Substituting equation (481) into $\boldsymbol{\nabla}\!\cdot\!\boldsymbol{v} = 0$ yields

$$\frac{\cos\theta}{r^2}\frac{\partial(r^2 A)}{\partial r} - \frac{B}{r\sin\theta}\frac{\partial(\sin^2\theta)}{\partial\theta} = 0 \tag{482}$$

which requires

$$B = \frac{1}{2r}\frac{d(r^2 A)}{dr}. \tag{483}$$

Note that the angular dependence of both terms in the divergence is the same (as it must be if our assumption for the form of $\boldsymbol{v}$ was correct), and that if $A \to 1$ at large r, then B will behave the same way, as required. We have,

$$v_r = VA\cos\theta, \qquad v_\theta = -\frac{V\sin\theta}{2r}\frac{d(r^2 A)}{dr}. \tag{484}$$

From these expressions, we may calculate the ϕ component of $\boldsymbol{\omega}$,

$$\omega_\phi = \frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right) = \frac{V\sin\theta}{r}\left[A - \frac{d^2}{dr^2}\left(\frac{r^2 A}{2}\right)\right] \tag{485}$$

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What we really need is the vector $\boldsymbol{\omega} = \omega_\phi \boldsymbol{e_\phi}$. Using the Cartesian representation for $\boldsymbol{e_\phi} = (-\sin\phi, \cos\phi, 0)$, $\omega_\phi \boldsymbol{e_\phi}$ has components

$$\boldsymbol{\omega} = V \left[\frac{A}{r} - \frac{1}{r}\frac{d^2}{dr^2}\left(\frac{r^2 A}{2}\right) \right] (-\sin\theta\sin\phi, \sin\theta\cos\phi, 0). \qquad (486)$$

Each of the nonvanishing Cartesian components must individually satisfy the Laplace equation, and we are in luck: both $\sin\theta\cos\phi$ and $\sin\theta\sin\phi$ are indeed the correct angular dependence of elementary solutions of this equation. $\sin\theta\cos\phi$ and $\sin\theta\sin\phi$ are both solutions of the Laplace equation when each is multiplied by a function of r that is a simple linear superposition of r and $1/r^2$. Hence, the term in square brackets must be proportional to this linear superposition of r and $1/r^2$. But we may immediately rule out the solution proportional to r, since it clearly would require A to grow at large r, whereas A must approach one. The $1/r^2$ function is acceptable for both Cartesian components of $\boldsymbol{\omega}$, hence we require

$$\left[A - \frac{d^2}{dr^2}\left(\frac{r^2 A}{2}\right) \right] = \frac{\text{const.}}{r}, \qquad (487)$$

where the constant on the right is as yet undetermined.

Equation (487) is easily solved. The general solution consists of a superposition of the solution to the homogeneous equation (A_h) plus a particular solution (A_p) to the inhomogeneous equation. The latter is obviously a constant times $1/r$. For A_h, note that the equation must have simple power law solutions. These are found to be $A_h = $ constant, and $A_h \propto 1/r^3$. Since $A \to 1$ at large r, we have:

$$A = 1 + \frac{\beta}{r^3} + \frac{\gamma}{r} \qquad (488)$$

where β and γ are constants to be determined. The corresponding B function is

$$B = \frac{1}{2r}\frac{d(r^2 A)}{dr} = 1 - \frac{\beta}{2r^3} + \frac{\gamma}{2r} \qquad (489)$$

Both A and B must vanish at the radius of the sphere $r = a$. This is sufficient to determine β and γ, and one finds

$$A = 1 + \frac{a^3}{2r^3} - \frac{3a}{2r}, \quad B = 1 - \frac{a^3}{4r^3} - \frac{3a}{4r} \qquad (490)$$

This completes are solution for v_r and v_θ:

$$v_r = V\cos\theta\left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right), \quad v_\theta = -V\sin\theta\left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right). \qquad (491)$$

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There remains the pressure P. Since $\nabla^2 P = 0$, it makes sense for this linear problem to search for a solution with the same periodicity as the velocity. There is only one such solution to the Laplace equation that is independent of ϕ and approaches a uniform pressure at infinity:

$$P = P_\infty + \frac{\alpha \cos \theta}{r^2} \tag{492}$$

where P_∞ is the pressure at infinity and α is a constant to be determined.

Determining α is the most difficult part of this problem. Though it can be done by using just one component of the Navier-Stokes equation, care must be taken. To avoid the complications of operating with the Laplacian on the unit vectors, we will take the z component:

$$0 = -\frac{\partial P}{\partial z} + \eta \nabla^2 v_z \tag{493}$$

Now,

$$\frac{\partial}{\partial z} = \boldsymbol{e_z} \cdot \boldsymbol{\nabla} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \tag{494}$$

and

$$v_z = \cos \theta v_r - \sin \theta v_\theta = V(A \cos^2 \theta + B \sin^2 \theta) \tag{495}$$

We find for the pressure gradient,

$$\frac{\partial P}{\partial z} = \frac{\alpha}{r^3}(1 - 3\cos^2 \theta) \tag{496}$$

while for v_z:

$$\frac{v_z}{V} = \left(1 + \frac{a^3}{2r^3} - \frac{3a}{2r}\right)\cos^2 \theta + \left(1 - \frac{a^3}{4r^3} - \frac{3a}{4r}\right)\sin^2 \theta \tag{497}$$

Before we take the Laplacian of v_z/V, note the identities

$$\nabla^2(1/r) = \nabla^2[(3\cos^2 \theta - 1)/r^3] = \nabla^2[(2\cos^2 \theta - \sin^2 \theta)/r^3] = 0 \tag{498}$$

since the first is the "point charge" solution of the Laplace equation, and the latter are two different forms of the quadrupole solution. Of course, the Laplacian of any constant vanishes trivially.

With the above identities, it is found that all constant and $1/r^3$ terms vanish with the application of ∇^2, and only the terms proportional to $1/r$ remain. These are

$$-\frac{3a}{2r}\left(\cos^2 \theta + \frac{\sin^2 \theta}{2}\right) = -\frac{3a}{4r}\left(1 + \cos^2 \theta\right) \tag{499}$$

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The term $-3a/4r$ vanishes when hit with ∇^2, leaving us with

$$-\frac{3a}{4r}\cos^2\theta = -\frac{3a}{4r}(\cos^2\theta - 1/3) - \frac{a}{4r} \qquad (500)$$

Again, we ignore the final $a/4r$ term, since it vanishes with the Laplacian operation. In the end, we are left with evaluating

$$-(3a/4)\nabla^2[(\cos^2\theta - 1/3)/r]. \qquad (501)$$

The radial derivatives of ∇^2 vanish (since they operate on $1/r$), and $\cos^2\theta - 1/3$ is a quadrupole eigenfunction of the angular derivatives. The operation

$$\nabla^2_{\text{angular}}(\cos^2\theta - 1/3) \equiv \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(\cos^2\theta - 1/3)\right)$$

returns $-6/r^2 \times (\cos^2\theta - 1/3)$, i.e. multiplication by a factor of $-l(l+1)/r^2$, which is expected for this $l = 2$ Legendre polynomial. Putting everything together, we find that $\eta\nabla^2 v_z =$

$$\frac{-6\eta}{r^2} \times \left(\frac{-3aV}{4r}\right)[\cos^2\theta - \frac{1}{3}] = \frac{3a\eta V}{2r^3}(3\cos^2\theta - 1) = \frac{\partial P}{\partial z} = -\frac{\alpha}{r^3}(3\cos^2\theta - 1) \qquad (502)$$

or

$$\alpha = -\frac{3a}{2}\eta V \qquad (503)$$

The pressure is

$$P = P_\infty - \frac{3a\eta V \cos\theta}{2r^2} \qquad (504)$$

This completes the determination of the pressure and the velocity for low Reynolds number flow around a sphere.

7.6.2 The drag force.

The stress tensor T_{ij} written in Cartesian form is

$$T_{ij} = -P\delta_{ij} + \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \qquad (505)$$

In spherical coodinates, there are additional terms originating with the derivatives of unit vectors, but these terms are all proportional to the velocity. We shall be evaluating the stress at the surface of the sphere, where the velocity

vanishes. Hence these additional terms may be ignored, and we may use this Cartesian form, even for our spherical problem. Moreover, the reader may verify that $\partial v_r / \partial r$ and $\partial v_r / \partial \theta$ both vanish at the sphere's surface.

To calculate the force exerted by the fluid on the sphere, recall that the stress tensor is the momentum flux at a given point and that local force on a differential surface area dS_i is then $T_{ij} dS_i$, where we sum over the repeated index i, as usual. This gives the force in the j direction.

The above considerations imply that the local radial force per unit area exerted by the fluid on the sphere is

$$F_r = T_{rr} = -P = -P_\infty + \frac{3\eta V \cos\theta}{2a} \tag{506}$$

and the local θ force is

$$F_\theta = T_{r\theta} = \eta \frac{\partial v_\theta}{\partial r} = -\frac{3\eta V \sin\theta}{2a} \tag{507}$$

The *net* force must be in the direction of the oncoming flow, that is, along the z axis:

$$F_z = \cos\theta F_r - \sin\theta F_\theta = -P_\infty \cos\theta + \frac{3\eta V}{2a} \tag{508}$$

The pressure at infinity has no effect when integrated over the surface area of the sphere, and the last term is constant. The net drag force is then simply

$$F_{drag} = \frac{3\eta V}{2a} \times 4\pi a^2 = 6\pi \eta V a \tag{509}$$

This is the classical formula derived by Stokes in 1845.

Exercise. Show that a solid sphere of density ρ_s will drift through a viscous liquid of density ρ_l and kinematic viscosity ν in a gravitational field g at a terminal velocity of

$$v = \frac{2a^2 g}{9\nu} \left(\frac{\rho_s}{\rho_l} - 1 \right).$$

Calculate this number for a steel sphere ($\rho = 8$ g cm^{-3}, $a = 1$ cm) in glycerine at $0°$ C ($\eta = \rho_l \nu = 120$ p, $\rho_l = 1.26$ g cm^{-3}). Here, 120 p means 120 poise, which is the cgs unit of dynamic viscosity. The cgs unit of *kinematic* viscosity is one stoke.

Exercise. The Stokes formula can be used to determine Avogadro's number, $N = 6.023 \times 10^{23}$, the number of molecules in one mole of gas! The result is due to Einstein, from his study of Brownian motion. Consider a tiny sphere

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of radius $a \sim 10^{-4}$ cm, which is still huge compared with a molecule. (In practice, these spheres were grains of pollen.) The sphere is in water, and subject both to the random forcing of the water molecules and to the viscous Stokes drag force. In this way, the sphere acquires a random kinetic energy $kT/2$ in one dimension. (Remember that $k = R/N$, where R is the classical gas constant 8.3 J per K-mole, and N is Avogadro's number.) Show that the sphere's motion in one dimension x satifies the equation

$$m\ddot{x} + 6\pi a\eta\dot{x} = f(t)$$

where m is the mass of the sphere and $f(t)$ is a rapidly varying random function of time t with zero mean value. Multiply this equation by the displacement x and show that the long term average displacement of the sphere satisfies

$$\langle x^2 \rangle = (kT/3\pi a\eta)t.$$

(Hint: $m\langle \dot{x}^2 \rangle = kT$.) Estimate $\langle x^2 \rangle^{1/2}$ for $t = 100$ seconds. Explain how to determine Avogadro's number with an ordinary microscope, a watch, and a grain of pollen. Note that as $t \to \infty$, the rms displacement becomes infinite, whereas the systematic velocity is zero.

7.6.3   Self-consistency

There is a potential problem with the result of equation (509). We have neglected the terms $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ in our analysis in favor of $\nu\nabla^2\boldsymbol{v}$. But if we now go back and check whether this is self-consistent, we find that the assumption breaks down at large radii, $r > \nu/U$. One might argue that since the trouble is far from the sphere it can't be important, but in fact making a small change at large distances can, in principle, have much larger consequences at small radii near the sphere.

The problem was not fully resolved until 1957, when Proudman and Pearson applied what is known as *matched asymptotics* to the problem. A detailed discussion would take us too far afield, but here is the idea.

When the Reynolds number is very small, the solution near the sphere can be treated with the neglected $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ as a small perturbation. What does "near the sphere mean?" It turned out that the solution obtained was valid provided that $r/a \ll 1/\mathcal{R}^{1/2}$ as $\mathcal{R} \to 0$. So, in fact, the solution is valid rather far away from the sphere, but not at infinite distances.

Another solution was found to be valid at large r. What does "large" mean? Large in this case meant $r/a \gg 1$. But that means if, for example, $r \sim 1/\mathcal{R}^{1/3}$, *both* solutions would have to be valid at once! The existence of an intermediate asymptotic zone satisfying both restrictions is a powerful

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constraint, because the two different solutions ($r/a \ll 1/\mathcal{R}^{1/2}, r/a \gg 1$) must be identical where they overlap. Thus, although the entire solution cannot be found at once, it can be found in pieces, and then patched together. All integration constants are fixed by this matching prescription.

What is the outcome of this mathematical procedure? Rest assured. Stokes's drag force is only modified by a factor of $1 + 3\mathcal{R}/8$ when $\mathcal{R}$ is small. Our relatively simple solution captures the essence of the problem nicely. The existence of matched asymptotic region not only allows the problem to be solved precisely, it also indicates why our original solution is as good as it is. What we have called the "near" solution actually extends quite far out, and its region of validity gets larger and larger at smaller and smaller Reynolds number. Though it is technically invalid at arbitrarily large distances, it "knows" about radii very far from the sphere $r = a$.

7.7 Viscous Flow Around Obstacles

This section contains optional advanced material, included for your interest.

When a viscous fluid passes over a smooth surface around an obstacle in its path, if the deflection does not occur at too steep an angle, the flow remains smooth and follows the contours of the obstacle. If the obstacle makes a steep angle with respect to the flow, however, something interesting happens. Where the obstacle rises from the smooth surface, the intersection, or "corner," that is formed is characterized by the formation of embedded vortices. The question is at what angle can a viscous flow be deflected before it starts to form internal vortices?

We can simplify our problem by looking at the very local properties of the flow. Consider viscous flow in a wedge of opening angle 2α. We use cylindrical coordinates R and ϕ; the flow is assumed to be independent of z. For $\phi > 0$, the radial velocity v_R is inwards, for $\phi < 0$ it is outwards. By contrast, the azimuthal velocity v_ϕ is symmetric with respect to ϕ. As usual, no-slip boundary conditions are enforced at $\phi = \pm\alpha$.

Let us begin by considering solutions of the form

$$\boldsymbol{v} = \boldsymbol{e_R}A(R)\cos(p\phi) + \boldsymbol{e_\phi}B(R)\sin(p\phi) \tag{510}$$

where A and B are to be determined. Clearly, this solution is compatible with the symmetry of the problem, and the general solution will be a Fourier superposition of such functions, since the governing equations are linear in $\boldsymbol{v}$. Mass conservation

$$\frac{1}{R}\frac{\partial(Rv_R)}{\partial R} + \frac{1}{R}\frac{\partial v_\phi}{\partial \phi} = 0 \tag{511}$$

yields immediately

$$B = \frac{1}{p}\frac{d(RA)}{dR} \tag{512}$$

The vorticity $\boldsymbol{\varpi}$ has only a z component for this problem, which we denote ϖ. As is familiar from the last section, the equation of motion for a viscous fluid is

$$\nabla^2 \varpi = 0, \tag{513}$$

so that a solution proportional to $\cos p\phi$ or $\sin p\phi$ must be of the form $R^p \cos p\phi$ (or $\sin p\phi$). The general solution is a superposition of solutions of this form with p positive or negative. If we next evaluate

$$\varpi = \frac{1}{R}\left[\frac{\partial(Rv_\phi)}{\partial R} - \frac{\partial v_R}{\partial \phi}\right] \tag{514}$$

we obtain

$$\varpi = \frac{\cos p\phi}{pR}\left(\frac{d}{dr}\left(R\frac{d(RA)}{dR}\right) - p^2 A\right) \tag{515}$$

This is consistent with our Laplace equation solution provided that

$$\frac{1}{pR}\left[\frac{d}{dr}\left(R\frac{d(RA)}{dR}\right) - p^2 A\right] = \text{constant} \times R^p \tag{516}$$

This is readily solved:

$$A(R) = C_1 R^{p+1} + C_2 R^{p-1} \tag{517}$$

where the first term is the particular solution to the inhomogeneous equation, and the second is a solution to the homogeneous equation. (Remember that p could be positive or negative.)

Thus far, we have grouped together all the terms proportional to $\cos(p\phi)$ or $\sin(p\phi)$. To implement the boundary conditions, which depend only upon ϕ, it is better to group together all of the trignometric terms that have the same power-law dependence in R. Notice that each trig function of $p\phi$ is associated with one power of p higher in R and one power of p lower. Thus, we consider a solution of the form

$$v_R = R^{p-1}\left(\beta_1 \sin[(p-2)\phi] + \beta_2 \sin(p\phi)\right) \tag{518}$$

Mass conservation $\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0$ then gives

$$v_\phi = R^{p-1}\left(\frac{p\beta_1}{p-2}\cos[(p-2)\phi] + \beta_2 \cos(p\phi)\right) \tag{519}$$

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The no-slip boundary conditions at $\phi = \pm\alpha$ are satisfied if

$$\beta_1 \cos(p\alpha) + \beta_2 \cos[(p-2)\alpha] = 0 \tag{520}$$

$$p\beta_1 \sin(p\alpha) + (p-2)\beta_2 \sin[(p-2)\alpha] = 0 \tag{521}$$

A solution is possible only if

$$p\tan(p\alpha) = (p-2)\tan[(p-2)\alpha] \tag{522}$$

This is equivalent to

$$\frac{\sin x}{x} = -\frac{\sin 2\alpha}{2\alpha}, \quad x = 2(p-1)\alpha \tag{523}$$

which is most easily derived by replacing the tangent functions in terms of their representation in terms of complex exponentials $e^{ip\alpha}$, etc. In this form, it is easy to see when there are real solutions for p and when there are not.

Plot the function $(\sin x)/x$. Note that it has a minimum value of -0.217 at $x = 4.4934$ radians. Thus, if $\sin(2\alpha)/2\alpha$ is small, less than 0.217 there is no difficulty finding a solution. But this implies a rather large angle for α, in excess of $146.3°$. In other words, α cannot be very far from an open $180°$ straight line. If α is less than $146.3°$, there are no real solutions for p. There are, however, complex-valued solutions for p. Since our fundamental equations are real and linear, the real and imaginary parts of $\boldsymbol{v}$ will each satisfy the equations separately. When p is complex, R^p becomes infinitely oscillatory as $R \to 0$, and along a ray of fixed ϕ, this suggests a sequence of smaller and smaller eddies. The formation of these eddies is in fact seen in the laboratory, though the amplitudes become sufficiently small that only first two can be measured.

7.8   Theory of Thin Films

One of the most important applications of viscous flows occurs when the fluid is trapped between two rigid boundaries at $z = 0$ and $z = h(x, y)$. (In principle, the lower boundary could also depend upon x and y, but here we restrict ourselves to this simpler case.) Lubrication theory, for example, involves the study of a viscous fluid between two very closely spaced surfaces, such as the cylinder wall and piston of an automobile engine. If L is a characteristic horizontal length scale, we assume that $h \ll L$. If U is a typical value of the v_x or v_y scale, then since

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0, \tag{524}$$

we find that

$$v_z \sim \frac{h}{L}U \ll U. \tag{525}$$

Vertical velocities are thus very small compared with v_x or v_y and will be ignored to leading order.

Because of the no-slip boundary conditions, the vertical gradient of horizontal velocities will be $\sim U/h$, much larger than the U/L scaling for the horizontal gradients. Hence, to an excellent approximation, for horizontal $\boldsymbol{v}$

$$\nu\nabla^2\boldsymbol{v} \simeq \nu\frac{\partial^2\boldsymbol{v}}{\partial z^2}, \tag{526}$$

the error being of order h^2/L^2. Because of the great difference in horizontal and vertical scales, the inertial term in the equation of motion $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ can be negligible even if the Reynolds number $\mathcal{R} \sim UL/\nu$ is large. In fact the condition

$$(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} \ll \nu\frac{\partial^2\boldsymbol{v}}{\partial z^2} \tag{527}$$

requires only that

$$\mathcal{R} \ll (L/h)^2, \quad \text{not } \mathcal{R} \ll 1. \tag{528}$$

This is what makes thin film theory interesting: it applies to both very small as well as to very large Reynolds numbers. Moreover, the simplified equation of motion

$$\boldsymbol{\nabla}P = \eta\frac{\partial^2\boldsymbol{v}}{\partial z^2} \tag{529}$$

shows that to order h/L, the z dependence of P may be ignored. (Recall the $\eta = \rho\nu$.) The horizontal equations of motion may then be immediately integrated with respect to z:

$$v_x = \frac{1}{2\eta}\frac{\partial P}{\partial x}z^2 + Az + B \tag{530}$$

$$v_y = \frac{1}{2\eta}\frac{\partial P}{\partial y}z^2 + Cz + D \tag{531}$$

where the pressure gradients and A, B, C, D all could depend upon both x and y.

7.8.1 The Hele-Shaw Cell

The simplest application of the above equations is to the problem where h is a constant, say the space between two plates of glass. The no-slip conditions give

$$v_x = \frac{1}{2\eta}\frac{\partial P}{\partial x}z(z-h) \tag{532}$$

$$v_y = \frac{1}{2\eta}\frac{\partial P}{\partial y}z(z-h) \tag{533}$$

Notice that since the horizontal velocity may be written as an exact nonsingular gradient, the circulation must always satisfy

$$\int \boldsymbol{v}\cdot\boldsymbol{dS} = 0 \tag{534}$$

Unlike two-dimensional "irrotational" flows in nature with hidden boundary layers, this so-called *Hele-Shaw flow* cannot develop a non-zero circulation integral by adding the equivalent of a line vortex $v_\phi = \Gamma/2\pi r$. Instead, the horizontal flow is exactly that of an inviscid flow with zero circulation and zero curl. This was Hele-Shaw's motivation for developing this experimental technique in 1898. Recall that two-dimensional flow past an airplane wing must develop circulation to avoid singular points, as we have seen. In a Hele-Shaw cell, non-zero circulation is forbidden, and the flow struggles valiantly to accommodate the singularities of zero circulation! Inevitably, the thin flim approximation breaks down when the horizontal gradients become comparable to $1/h$, but it is an excellent assumption over all larger scales. See the striking photographs in Van Dyke, *An Album of Fluid Motion,* pp. 8-11.

7.9   Adhesive Forces

You have probably noticed that if you wet a rubber disk and stick it on a surface that it is very difficult to pull it away. The reason for this can be examined quantitatively.

Consider a viscous fluid between two surfaces that are separated by a distance $h(t)$, a function of time only. For all times of concern, $h \ll a$, where a is the radius of the upper surface. We shall assume, and verify *a posteriori,* that the explicit time derivative term $\partial \boldsymbol{v}/\partial t$ is small compared with $\nu\nabla^2\boldsymbol{v}$. Thus, hydrostatic equilibirum is always instantaneously maintained as the upper surface is lifted. (Any departures from hydrostatic equilibrium would be corrected on the very rapid time scale h^2/ν.) The time dependence appears only implicitly via the boundary conditions at $z = h(t)$.

What are the boundary conditions for this problem? The radial velocity v_R must vanish at $z = 0$ and $z = h(t)$ by the no-slip constraint. The vertical velocity v_z must vanish at $z = 0$, and follow the surface at $z = h(t)$:

$$v_z = dh/dt \quad \text{at } z = h(t). \tag{535}$$

The radial velocity is the exact analog of the Hele-Shaw planar velocity components:

$$v_R = \frac{1}{2\eta}\frac{\partial P}{\partial R}z(z - h) \tag{536}$$

where P depends only upon R and t. The vertical velocity is given by the integrated form of the mass conservation equation, with $v_z = 0$ at $z = 0$:

$$v_z = -\int_0^z \frac{1}{2\eta R}\frac{\partial}{\partial R}\left(\frac{R\partial P}{\partial R}\right) z(z - h)\,dz = -\frac{1}{2\eta R}\frac{\partial}{\partial R}\left(\frac{R\partial P}{\partial R}\right)\left(\frac{z^3}{3} - \frac{z^2 h}{2}\right) \tag{537}$$

Applying the boundary condition $v_z = dh/dt$ at $z = h$ gives

$$\frac{\partial}{\partial R}\left(\frac{R\partial P}{\partial R}\right) = \frac{12\eta R}{h^3}\frac{dh}{dt}, \tag{538}$$

which is valid everywhere since the pressure is independent of z, and a vertical velocity of

$$v_z = \frac{dh}{dt}\zeta^2(3 - 2\zeta), \tag{539}$$

where $\zeta(t) = z/h(t)$. The pressure may be found from integrated equation (538). Integrating once yields

$$\frac{\partial P}{\partial R} = \frac{6\eta R}{h^3}\frac{dh}{dt} + \frac{C(t)}{R} \tag{540}$$

Clearly, $C(t)$ must vanish since the pressure is nonsingular at the origin. From equation (536), the radial velocity is

$$v_R = 3\zeta(\zeta - 1)\,R\frac{d\ln h}{dt} \tag{541}$$

which is, as expected, larger than v_z by a factor of R/h. (Notice that it is negative, or inward, if $dh/dt > 0$.) The pressure is obtained from a final integration of equation (540):

$$P = \frac{3\eta}{h^3}\frac{dh}{dt}(R^2 - D(t)) \tag{542}$$

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where $D(t)$ is a time-dependent integration constant. Taking an atmospheric pressure of zero at $R = a$ then gives

$$P = \frac{3\eta}{2h^3}\frac{dh}{dt}(R^2 - a^2) \tag{543}$$

The force exerted by the fluid on the upper surface is

$$2\pi \int_0^a P\,R\,dR = -\frac{3\pi\eta a^4}{h^3}\frac{dh}{dt} \tag{544}$$

This is negative ($dh/dt > 0$), therefore adhesive, and in general very large, especially for an extended upper surface and a tight seal ($a^4/h^3 \ll 1$).

The last task is to justify the neglect of the $\partial/\partial t$ term in the equation of motion. If we compare this with $\nu\nabla^2$, and use

$$\partial/\partial t \sim (d\ln h/dt), \quad \nabla^2 \sim (1/h^2), \quad v_R \sim a(d\ln h/dt), \tag{545}$$

we find that the criterion for neglecting the time derivative is identical to the Reynolds number criterion (528).

Exercise. At fixed R, compute the z-averaged value for v_R. Show that at $R = a$, this average value is $-(a/2)d\ln h/dt$. Interpret this physically. (Hint: think in terms of mass conservation.)

8 Boundary Layers

A smooth flow with no solid boundaries generally makes the transition from inviscid $\nu = 0$) to viscous (finite ν) continuously. When the viscosity is small, it introduces small perturbations throughout the flow. When a bounding wall is present, however, the no-slip boundary conditions introduce themselves *discontinuosly*: the introduction of an arbitrarily small viscosity changes the flow by a finite amount near the wall. What changes continuously as a very small ν is introduced is not the velocity field of the flow, but the thickness of the layer over which there is a finite velocity change. This layer, which defines the region in which the no-slip boundary conditions have changed the velocity by "order unity" compared with inviscid flow, is known as the *boundary layer*. Boundary layer theory remains an active area of fluid research.

The simplest manifestation of a mathematical boundary layer is provided by the equation
$$\epsilon y' + y = 1, \quad y(0) = 0, \quad \epsilon \to 0. \tag{546}$$

When $\epsilon = 0$ exactly, the solution is $y = 1$, end of story. The boundary condition cannot be satisfied. The exact solution is

$$y = 1 - e^{-x/\epsilon} \tag{547}$$

As $\epsilon \to 0$, it is not possible to find a solution of the form $y(x; \epsilon) = y_0(x) + \epsilon y_1(x)$. The limit $\epsilon \to 0$ of $y(x; \epsilon)$ is an essential singularity in ϵ and has no Taylor Series expansion.

Note that it matters in what order the limits $x \to 0$ and $\epsilon \to 0$ are taken. In boundary layer theory, we always take the ϵ parameter to be small but finite, letting it tend to zero only at the very end of the calculation.

If, however, we define a new variable, the so-called *inner variable* $X = x/\epsilon$, the equation becomes

$$\frac{dy}{dX} + y = 1, \quad y(0) = 0 \tag{548}$$

and the boundary layer appears to have vanished! This too is typical of boundary layer problems. When the independent variable is rescaled in this way, the inner solution becomes a non-singular differential equation as $\epsilon \to 0$.

Here is an interesting example with a second order differential equation:

$$\epsilon y'' + a(x)y' + b(x)y = 0, \quad y(0) = A, \ y(1) = B \tag{549}$$

It is not possible to solve this problem analytically for arbitrary $a(x)$ and $b(x)$, but we *can* obtain a general solution in the limit $\epsilon \to 0$. Let us assume that there is a boundary layer near $x = 0$. Then, our outer equation is

$$ay_o' + by_o = 0 \tag{550}$$

which has the solution

$$y_o = B \exp\left[\int_x^1 b(t)/a(t) \, dt \right] \tag{551}$$

We have used the boundary condition at $x = 1$, but not the boundary condition at $x = 0$, which is part of the inner solution in the boundary layer. The use of only one boundary condition is mathematically consistent with the equation becoming first order. (We require that $a(x)$ does not vanish over the interval $[0, 1]$.)

Inside the boundary layer we expect the solution to be rapidly varying, so that $y \ll y'$. Our inner equation is taken very near $x = 0$, so we may replace $a(x)$ by $\alpha = a(0)$:

$$\epsilon y_i'' + \alpha y_i' = 0. \tag{552}$$

which has the solution

$$y_i = A + C\left[\exp(-\alpha x/\epsilon) - 1\right]. \tag{553}$$

C is an integration constant, and we have used the boundary condition $y(0) = A$. To have a self-consistent boundary layer, we require $\alpha > 0$, and since $a(x)$ cannot change sign, $a(x) > 0$ over the interval $[0, 1]$.

The trick now is to note that the inner solution remains valid when x/ϵ is large, provided that x itself small! For example, if $\exp(-\alpha x/\epsilon)$ is of order $\epsilon^{1/2}$, all of the approximations of the inner zone are still valid as $\epsilon \to 0$. This corresponds to $x \sim |\epsilon \ln \epsilon| \gg \epsilon$. Since $x \ll 1$, we may still replace the functions by their values at $x = 0$, and we are still justified in neglecting the final term in the equation, by, since it is smaller than the other terms by a factor of order $\epsilon^{1/2}$. In other words, our inner solution is also valid just outside the boundary layer of rapid change, provided that we keep x small.

The outer solution, on the other hand, remains valid provided that we stay away from the inner boundary layer of rapid change. The outer solution certainly can, in principle, remain valid in a region where the inner solution also happens to be valid. Indeed, we can see that there will be a region of small x but large x/ϵ where *both* the inner and outer solutions must be simultaneously valid, and they must give the same answer! For large x/ϵ,

$$y_i \to A - C, \tag{554}$$

and for small x

$$y_o \to B \exp\left[\int_0^1 b(t)/a(t)\,dt\right] \tag{555}$$

These two solutions will agree provided that

$$C = A - B \exp\left[\int_0^1 b(t)/a(t)\,dt\right] \tag{556}$$

or

$$y_i = Ae^{-\alpha x/\epsilon} + Be^{\int_0^1 (b/a)dt}\left(1 - e^{-\alpha x/\epsilon}\right). \tag{557}$$

Finally, we may construct a single expression that is valid *everywhere*:

$$y = y_o + y_i - B \exp\left[\int_0^1 b(t)/a(t)\,dt\right] \tag{558}$$

This works in the inner region because y_o is canceled out by the final term. This works in the outer region because y_i is canceled out by the final term. And this works in the matching region, because all three terms are the same,

and twice the solution minus the solution is the solution! Our final expression for y is therefore:

$$y = \left(A - Be^{\int_0^1 (b/a)\, dt} \right) e^{-\alpha x/\epsilon} + Be^{\int_x^1 (b/a)\, dt} \tag{559}$$

This completes our solution to the differential equation. We were able to obtain an analytic solution to this problem in the limit of small ϵ, essentially because the problem breaks up into two solvable differential equations: one of lower order, the other with locally constant coefficients. The matched asymptotic expansion then joined the two separately obtained solutions smoothly. (See Bender & Orszag, *Advanced Mathematical Methods for Scientists and Engineers,* for details of how to carry this to higher order accuracy.)

Let us see how our mathematics applies to high Reynolds number fluids near boundary layers.

8.1 The Boundary Layer Equations

Consider the flow of a viscous fluid across a planar surface, $y = 0$. In this section and the next, y will indicate the *vertical* direction. The horizontal direction is x. There is a thin boundary layer near $y = 0$ where the flow changes rapidly to satisfy the no-slip boundary conditions. The fundamental fluid equations for steady flow are

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} \right) \tag{560}$$

$$v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} \right) \tag{561}$$

along with mass conservation:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{562}$$

Boundary layer theory enjoys some of the simplifications we found in thin film theory. Boundary layers are also thin, in the sense that the characteristic length scale in the x direction (L), is much larger than the boundary layer thickness and characteristic scale in the y direction (δ). The mass conservation equation implies

$$v_y \sim \frac{\delta}{L} v_x \ll v_x, \tag{563}$$

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an important simplification. From this we may deduce two more simplifications: (i) $\partial P/\partial y \ll \partial P/\partial x$, which follows from comparing the y equation of motion with the x equation; and (ii) $\partial^2/\partial y^2 \gg \partial^2/\partial x^2$ in all viscous terms. These simplifications mean that there are two equations to be solved for v_x and v_y,

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = -\frac{1}{\rho}\frac{dP}{dx} + \nu \frac{\partial^2 v_x}{\partial y^2} \tag{564}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{565}$$

with dP/dx smoothly continuous through the boundary layer, given by its value in the inviscid flow just adjacent to the boundary layer. This is our fundamental set of equations.

The scale for the thickness of the boundary layer δ is obtained by setting

$$v_x \frac{\partial v_x}{\partial x} \sim \nu \frac{\partial^2 v_x}{\partial y^2} \tag{566}$$

This gives

$$\frac{\delta}{L} \sim \left(\frac{\nu}{v_x L}\right)^{1/2} \sim \mathcal{R}^{-1/2} \tag{567}$$

where L is a typical x scale and $\mathcal{R}$ is the Reynolds number of the flow. In the case of flow passing over a semi-infinite plane whose edge is at $x = 0$, the "typical x scale" L is just x itself, the horizontal distance from the edge. We then expect the thickness of the boundary layer to be a function of x, $\delta \sim \sqrt{\nu x}$. The viscous boundary layer grows as we move farther into the region $x > 0$, because viscous diffusion (really vorticity diffusion) has had more time to establish itself through the body of the flow. In the absence of external forces, any fluid element, no matter how distant from the plane, would eventually come to rest, as the no-slip boundary condition is propagated through the flow. Let us see how this works in detail.

8.2   Boundary Layer Near a Semi-Infinite Plate

Consider steady flow in the semi-plane $y > 0$. At the origin, a surface at $y = 0$, $x \geq 0$ enforces no-slip boundary conditions in the flow immediately adjacent to it. Before the flow encounters this surface, it moves with constant $v_x = U$ and $v_y = 0$, and there is no pressure gradient. What is the flow profile in the region $x \geq 0$?

The comments at the end of the last section suggest looking for a solution in which the velocities depend upon the combination y/δ, where $\delta = \sqrt{\nu x/U}$.

With the advantage of hindsight, it proves convenient to insert a factor of 2. Define

$$\eta = y \, (U/2\nu x)^{1/2} \, . \tag{568}$$

Then

$$\frac{\partial}{\partial y} = \frac{\eta}{y} \frac{d}{d\eta}, \quad \frac{\partial}{\partial x} = -\frac{\eta}{2x} \frac{d}{d\eta}. \tag{569}$$

We search for a solution of the form $v_x = UF(\eta)$. To satisfy the equation of mass conservation, we introduce the stream function ψ:

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x} \tag{570}$$

Integration of the first of these gives

$$\psi = (2\nu x/U)^{1/2} \int_0^\eta UF \, d\eta \tag{571}$$

where we have used the fact that ψ should be constant along the stream line $y = 0$. (Remember that ψ is constant along streamlines!). Hence, we define

$$f(\eta) = \int_0^\eta F \, d\eta \tag{572}$$

so that

$$\psi = (2U\nu x)^{1/2} f, \quad v_x = Uf' \tag{573}$$

and

$$v_y = -\frac{\partial \psi}{\partial x} = (U\nu/2x)^{1/2}(\eta f' - f) \tag{574}$$

where f' denotes $df/d\eta$. The equation of motion (564) is then remarkably concise:

$$f''' + ff'' = 0 \tag{575}$$

The boundary conditions are

$$f(0) = f'(0) = 0, \tag{576}$$

since v_x and v_y must vanish at $y = 0$, and

$$f'(\infty) = 1, \tag{577}$$

since the flow approaches $v_x = U$ at large y outside the boundary layer.

The boundary conditions at $\eta = 0$ suggest that $f \simeq A\eta^2$ near the origin, with A to be determined. Then,

$$f''' = -2A^2\eta^2, \tag{578}$$

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which would give the next term in the Taylor series as $-A^2\eta^5/24$. Evidently, f is very nearly quadratic near the surface. A numerical solution that satisfies the boundary condition at ∞ gives $A = 0.2348$.

The solutions near the $\eta = 0$ take the form

$$v_x = Uf' = 2UA\eta, \quad v_y = (U\nu/2x)^{1/2}(\eta f' - f) = (U\nu/2x)^{1/2}(A\eta^2) \quad (579)$$

A quantity of interest is the viscous stress

$$\frac{\rho\nu}{2}\frac{\partial v_x}{\partial y} = \rho\nu UA \left(\frac{U}{2\nu x}\right)^{1/2}, \tag{580}$$

the term $\partial v_y/\partial x$ being negligible by comparison. (This is in accord with the boundary layer approximations.) This is the x component of the force imparted to the plate per unit surface area (with normal in the y direction) by the flow. Let us assume that this formula, which is derived for a semi-infinite half-plane, is true for a very large surface of extent $L \gg (UL/2\nu)^{1/2}$. (Why? What does this mean?) Then the drag force per unit length is

$$2\int_0^L \rho\nu UA \left(\frac{U}{2\nu x}\right)^{1/2} dx = \rho 2A(2U^3 L\nu)^{1/2} \tag{581}$$

where the initial factor of 2 comes from two sides of the plate. This can be measured in the laboratory, and the agreement with theory is good. Note that the drag force is proportional only to $L^{1/2}$, not to L, because the velocity gradients diminish at large distances along the surface, and that the drag vanishes at large Reynolds numbers. In reality, it is found that the boundary layer becomes turbulent beyond $\mathcal{R} \sim 10^5$ or so, and the drag force does *not* vanish.

8.3   Ekman Layers

When a rotating fluid is confined from above by fixed walls, the resulting boundary layer, in which the rotation is brought to a halt, is known as an *Ekman* layer. Unlike the flows we have been studying until now, an Ekman layer can exert an influence throughout the entire flow even when it remains firmly attached to the surface and nonturbulent. Let us see how this comes about in a relatively simple example.

The Navier-Stokes equation in a frame rotating at angular rate Ω is

$$\frac{D\boldsymbol{v}}{Dt} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\boldsymbol{\nabla}H + \nu\nabla^2\boldsymbol{v} \tag{582}$$
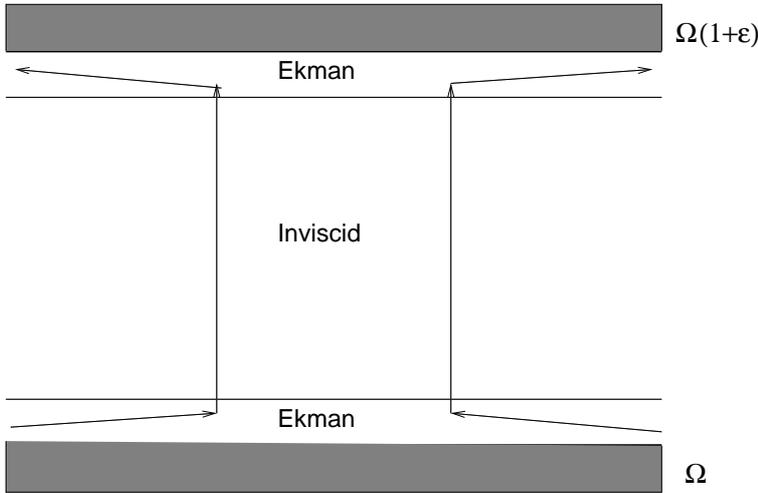
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Figure 15: A schematic diagram of the secondary flow induced by Ekman layers. The flow is nearly horizontal in the boundary layers, and nearly vertical in the inviscid interior. The flow moves toward the center at the bottom, where its rotation is slowed, and toward the exterior at the top, where the rotation is increased.

where D/Dt is the Langrangian derivative, and H is a sort of generalized enthalpy function:

$$dH = \frac{dP}{\rho} + d\Phi - d(R\Omega^2/2) \tag{583}$$

where Φ represents external potential forces (e.g., gravity), and the final term is the centrifugal force. We shall assume throughout our discussion of Ekman layers that the flow is dominated by rotation, i.e., $v \ll R\Omega$. Hence we ignore the inertial term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$ compared with the Coriolis force $2\boldsymbol{\Omega} \times \boldsymbol{v}$. For steady, inviscid flow, conditions for the Taylor-Proudman theorem hold, and the flow velocity $\boldsymbol{v}$ will be independent of z.

Consider the problem in which a rotating fluid has boundaries at $z = 0$ and $z = L$ that are rotating at angular velocities Ω and $\Omega(1+\epsilon)$ respectively. These boundary conditions are not consistent with the Taylor-Proudman theorem, since the rotation rate changes with height, and we expect boundary layers to form at one, or possibly both ends. (See figure [15].)

Away from the boundary layers, the inviscid equations are valid. We denote these quantities with the subscript I. In Cartesian coordinates,

$$-2\Omega v_{Iy} = -\frac{\partial H}{\partial x} \tag{584}$$

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$$2\Omega v_{Ix} = -\frac{\partial H}{\partial y} \tag{585}$$

while in cylindrical coordinates,

$$-2\Omega v_{I\phi} = -\frac{\partial H}{\partial R} \tag{586}$$

$$2R\Omega v_{IR} = -\frac{\partial H}{\partial \phi} = 0 \tag{587}$$

(The last equation holds since the flow is axisymmetric.) Hence, the interior flow depends only upon R, has no radial component, and as we shall see, is determined entirely by matching to the boundary layers above and below.

Within the very narrow boundary layers, the equations of motion in Cartesian coordinates are

$$-2\Omega v_y = -\frac{\partial H}{\partial x} + \nu \frac{\partial^2 v_x}{\partial z^2} \tag{588}$$

$$2\Omega v_x = -\frac{\partial H}{\partial y} + \nu \frac{\partial^2 v_y}{\partial z^2} \tag{589}$$

$$0 = -\frac{\partial H}{\partial z} + \nu \frac{\partial^2 v_z}{\partial z^2} \tag{590}$$

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \tag{591}$$

The final equation of mass conservation combined with the assumption that the boundary layer is very narrow implies that v_z is much smaller than v_x or v_y, by a factor of order δ/L where δ is the boundary layer thickness and L is a horizontal scale. The first three equations of motion then lead to the conclusion that H depends much more strongly upon x and y than upon z, and its vertical gradients may be ignored. If H does not depend upon z, its horizontal gradients are determined in the boundary layer by their value in the inviscid flow, equations (584) and (585). Therefore, the first two boundary layer equations of motion may be written

$$-2\Omega(v_y - v_{Iy}) = \nu \frac{\partial^2 v_x}{\partial z^2} \tag{592}$$

$$2\Omega(v_x - v_{Ix}) = \nu \frac{\partial^2 v_y}{\partial z^2} \tag{593}$$

A nice trick is to combine these two equations into a single complex-valued differential equation:

$$\nu \frac{\partial^2 (v_x + i v_y)}{\partial z^2} = 2\Omega i [(v_x - v_{Ix}) + i(v_y - v_{Iy})] \tag{594}$$

Since the v_I do not depend upon z, this is simply

$$\nu \frac{\partial^2 f}{\partial z^2} = 2\Omega i f \tag{595}$$

where

$$f = (v_x - v_{Ix}) + i(v_y - v_{Iy}) \tag{596}$$

This has the solution

$$f = A(x,y) \exp[(\Omega/\nu)^{1/2} z(1+i)] + B(x,y) \exp[-(\Omega/\nu)^{1/2} z(1+i)] \tag{597}$$

where A and B are functions of x and y. We introduce the scaled z variable

$$Z = z(\Omega/\nu)^{1/2} \tag{598}$$

The thickness of the boundary layer is thus of order $(\nu/\Omega)^{1/2}$.

Consider first the boundary condition at $z = Z = 0$. We shall assume that the surface rotates at a rate Ω_0 relative to the frame rotating at Ω. We will later set this equal to zero, but let us keep it general for the moment. Since the function f must vanish as $Z \to \infty$, $A = 0$. The function B is then determined by the $Z = 0$ boundary condition of solid body rotation

$$f = B(x,y) = (-y\Omega_0 - v_{Ix}) + i(x\Omega_0 - v_{Iy}) \tag{599}$$

This implies

$$(v_x - v_{Ix}) + i(v_y - v_{Iy}) = e^{-Z(1+i)} [(-y\Omega_0 - v_{Ix}) + i(x\Omega_0 - v_{Iy})] \tag{600}$$

If we sort this equation into its real and imaginary parts, we obtain

$$v_x = v_{Ix} + e^{-Z} [(x\Omega_0 - v_{Iy}) \sin Z - (y\Omega_0 + v_{Ix}) \cos Z] \tag{601}$$

$$v_y = v_{Iy} + e^{-Z} [(y\Omega_0 + v_{Ix}) \sin Z + (x\Omega_0 - v_{Iy}) \cos Z] \tag{602}$$

The vertical velocity is obtained from

$$-\frac{\partial v_z}{\partial z} \equiv -\left(\frac{\Omega}{\nu}\right)^{1/2} \frac{\partial v_z}{\partial Z} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}\right) \tag{603}$$

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Since $(\partial v_{Ix}/\partial x) + (\partial v_{Iy}/\partial y) = 0$ (why?), we find

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = e^{-Z} \sin Z \left(2\Omega_0 + \frac{\partial v_{Ix}}{\partial y} - \frac{\partial v_{Iy}}{\partial x} \right) = e^{-Z} \sin Z (2\Omega_0 - \omega_I) \quad (604)$$

where ω_I is the vorticity of the inviscid flow. The vertical velocity v_z at the outer edge of the Ekman layer is then

$$-\left(\frac{\Omega}{\nu} \right)^{1/2} v_z(out) = (2\Omega_0 - \omega_I) \int_0^\infty e^{-Z} \sin Z \, dZ \quad (605)$$

or,

$$v_z(out) = \left(\frac{\nu}{\Omega} \right)^{1/2} \left(\frac{\omega_I}{2} - \Omega_0 \right) \quad (606)$$

At the top Ekman layer, we do exactly the same calculation (with $\Omega = \Omega_L$, say). Now it is the B function that must be zero in equation (597), since Z is decreasing as we move from the boundary layer into the inviscid zone. We find exactly the same result (do it!), except for a minus sign:

$$v_z(out) = \left(\frac{\nu}{\Omega} \right)^{1/2} \left(\Omega_L - \frac{\omega_I}{2} \right) \quad (607)$$

We have used the same notation for $v_z(out)$ in the last two equations, because they must in fact be the same! The vertical velocity in the inviscid zone must join smoothly to the Ekman layers; it cannot be a function of z. With $\Omega_0 = 0$, and $\Omega_L = \epsilon\Omega$,

$$\epsilon\Omega - \frac{\omega_I}{2} = \frac{\omega_I}{2} \rightarrow \omega_I = \epsilon\Omega, \text{ and } v_z(out) = \left(\frac{\nu}{\Omega} \right)^{1/2} \frac{\epsilon\Omega}{2}. \quad (608)$$

Knowing the vorticity ω_I of the inviscid zone, it is a simple matter to find the angular velocity $v_{I\phi}$. It cannot depend upon z or ϕ, hence

$$\omega_I \equiv \frac{1}{R} \frac{d(Rv_{I\phi})}{dR} = \epsilon\Omega, \quad (609)$$

and

$$v_{I\phi} = \frac{\epsilon}{2} R\Omega \quad (610)$$

is the nonsingular solution.

To summarize: in a steady rotating flow in which end caps at $z = 0$ and $z = L$ rotate respectively at Ω and $\Omega(1 + \epsilon)\Omega$, there is a nearly inviscid

interior solution rotating at the average angular velocity of the two endcaps, and two Ekman layers at either end. Within the Ekman layers the flow is primarily in the horizontal plane, and satisfies no-slip boundary conditions. Emerging from the layers, there is a small (order δ/R) vertical flow from bottom to top that slowly mixes the upper and lower ends.

Exercise. Where have we used the fact that ϵ is small?

Exercise. Show that within the lower Ekman layer, the radial and azimuthal velocities are

$$v_R = -\frac{\epsilon}{2}R\Omega e^{-Z}\sin Z \tag{611}$$

$$v_\phi = \frac{\epsilon}{2}R\Omega(1 - e^{-Z}\cos Z) \tag{612}$$

Notice that v_R is directed inwards near the $Z = 0$ boundary: the decrease in the angular velocity near the inner no-slip boundary drains the rotating fluid of angular momentum, which then slowly drifts inwards. (Remember this as you read the next section.)

8.4   Why does a teacup slow down after it is stirred?

You've been working hard reading these notes, so make yourself a cup of tea and come right back.

Good, now that you've got a cup of tea in your hands, we can do some more fluid mechanics. Stir your tea, and notice how long it continues to rotate after you stop. It should be several seconds. Now, the kinematic viscosity of water is 0.01 cm^2 s^{-1}. If viscous diffusion were the sole reason that your tea slowed down, the associated time scale would be $\tau \sim R^2/\nu$. With $R = 2$ cm, this is a slow down time of 400 seconds, or almost 7 minutes! This is estimate is off by two orders of magnitude from the experimental result. Clearly, something else is going on here.

Here is a clue. Look at the bottom of your teacup. If you are using loose tea, you'll notice that the tea leaves have been pushed to the center. That is the key to the solution!

You may be feeling a bit like Dr. Watson after Sherlock Holmes announces that all is clear. It is time to do an analysis.

8.4.1 Slow down time for rotating flow.

Consider a flow rotating with angular velocity $\Omega(1 + \epsilon)$ in the presence of two endcaps at $z = \pm L/2$, each rotating at the same angular velocity Ω. We wish to study the induced flows and to calculate how long it takes for the flow to come to rest relative to the endcaps.

Very quickly, two time-steady Ekman layers will establish themselves at $z = \pm L/2$. This happens on a diffusion time scale across the narrow boundary layer itself, and this time scale is just $1/\Omega$. The bulk of the interior will continue, for the time being, to rotate at $\Omega(1 + \epsilon)$. Both of the endcaps are now rotating more *slowly* than the fluid, and the results of the last section show that there will be a vertical velocity of

$$|v_z| = \left(\frac{\nu}{\Omega}\right)^{1/2} \left(\frac{\omega_I}{2}\right) \tag{613}$$

emerging from each boundary layer. Here, as before, ω_I is the inviscid vorticity. Because the flow is now time-dependent, the Taylor-Proudman theorem now longer holds, and v_z may depend upon z, even in the inviscid interior. Indeed, it must, since v_z changes sign from one endcap to the other!

The equations of motion in the frame rotating at Ω of the inviscid interior are

$$\frac{\partial v_R}{\partial t} - 2\Omega v_\phi = -\frac{\partial H}{\partial R} \tag{614}$$

$$\frac{\partial v_\phi}{\partial t} + 2\Omega v_R = 0 \tag{615}$$

$$\frac{1}{R}\frac{\partial(Rv_R)}{\partial R} + \frac{\partial v_z}{\partial z} = 0. \tag{616}$$

Notice that we use cylindrical coordinates. (To keep the notation simple, we drop the subscript I on the velocities, but will retain it on ω_I.) Multiplying the azimuthal equation of motion by R and differentiating $\partial/\partial R$ gives

$$\frac{\partial}{\partial t}\left(\frac{\partial(Rv_\phi)}{\partial R}\right) = -2\Omega\frac{\partial(Rv_R)}{\partial R} = 2R\Omega\frac{\partial v_z}{\partial z} \tag{617}$$

where the last equality follows from mass conservation. This equation is equivalent to

$$\frac{1}{R}\frac{\partial}{\partial t}\left(\frac{\partial(Rv_\phi)}{\partial R}\right) \equiv \frac{\partial \omega_I}{\partial t} = 2\Omega\frac{\partial v_z}{\partial z} \tag{618}$$

Now, since the vertical velocity v_z is very small and by assumption $\Omega\tau \gg 1$, vertical hydrostatic equilibrium ($\partial H/\partial z \ll \partial H/\partial R$) is an excellent approximation. Therefore, we seek solutions in which neither v_R nor v_ϕ depend upon z. If we now integrate from $z = -L/2$ to $z = +L/2$, we obtain

$$L\frac{\partial \omega_I}{\partial t} = 2\Omega[v_z(L/2) - v_z(-L/2)] = -2(\nu\Omega)^{1/2}\omega_I \qquad (619)$$

With the initial condition of solid body rotation (at a rate of $\epsilon\Omega$ in the rotating frame), this equation has the simple solution

$$\omega_I = 2\epsilon\Omega e^{-t/\tau} \qquad (620)$$

where the spin down time τ is

$$\tau = \frac{L}{2(\nu\Omega)^{1/2}} \qquad (621)$$

This is clearly the time scale that is relevant for out tea cup problem, though a tea cup has sides, but no top. Notice that τ is the geometric mean of the viscous time L^2/ν, and the rotation time, $1/\Omega$. With $L = 2$ cm, $\Omega = 2\pi$, we obtain $\tau = 4$ s, which is a much better agreement with reality. Notice that it is the vertical height L that enters, even if the radius is infinite!

But we still don't really understand the physical process. Why is it so much faster than viscous dissipation? And what about the tea leaves?

Let us look at the induced radial and azimuthal flow components. Since

$$\omega_I = \frac{1}{R}\frac{\partial (Rv_\phi)}{\partial R} = 2\epsilon\Omega e^{-t/\tau} \qquad (622)$$

we find immediately that

$$v_\phi = R\epsilon\Omega e^{-t/\tau}, \qquad (623)$$

the solution that is finite at $R = 0$. This shows the decay of the dominant rotation velocity in excess of Ω. The azimuthal equation equation of motion immediately gives the radial velocity component,

$$v_R = -\frac{1}{2\Omega}\frac{\partial v_\phi}{\partial t} = \epsilon(\nu\Omega)^{1/2}(R/L)e^{-t/\tau}, \qquad (624)$$

while v_z follows immediately from integration of the mass conservation equation (616):

$$v_z = -2\epsilon(\nu\Omega)^{1/2}(z/L)e^{-t/\tau}, \qquad (625)$$

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Note that v_z depends upon z (and agrees with equation [613]), while v_R and v_ϕ depend only upon R and t.

We may now put together the whole picture. In the Ekman layer the radial velocity drifts inward (bringing the tea leaves along!), while the fluid is losing angular momentum to the sides of the cup. There is a tiny but macroscopic boundary layer of thickness $(\nu/\Omega)^{1/2} \sim 0.04$ cm at the bottom of the cup. Outside this boundary layer, the radial velocity is outward, and since this flow is inviscid, vorticity conservation lowers the angular velocity of a given fluid element. The fluid element eventually reaches the sides of the cup (at "infinity" in this calculation), moves downward, and then back along the bottom toward the center. In this return process, its vorticity is not conserved because of viscous effects. Expansion of the vortex in the inviscid interior is the slow-down mechanism in the bulk of the flow, and is the dynamical behavior that leads to the damping time τ in the equations. Notice how both dissipation (the presence of v_z) and vorticity conservation (outward v_R) play critical roles, which is why in the expression for τ they appear in a geometric mean.

What a marvelous, surprisingly rich problem this has been! It is well-posed, simple to describe, but remarkably subtle and requiring ingenuity to solve. It is a good example of why many people are fascinated by fluids.

9 Instability

9.1 Introduction

Take a glass of water, and fill it to the edge, perhaps even slightly over the edge (surface tension will help). Now take a piece of cardboard, a little larger than the area of the circular opening, and cover the glass. Keeping the cardboard firmly pressed against the glass, turn the glass over and slowly relax the tension you have been placing on the cardboard. You should find that the water in the glass happily remains inside, even without pressure on the cardboard cover. (By the way, do this over a sink.)

Why does the water remain in the glass? Well, why shouldn't it remain in the glass? After all, the atmosphere exerts a pressure on the water which is more than sufficient to prevent it from falling out. But fall out is just what the water does, if we so much as tap the cardboard even very lightly.

The piece of cardboard seems to be key this process. In fact, if we tried to do this without it, no matter how careful we were, we could not prevent the water from emptying out of the glass. This is not because the force of

gravity is unbalanced when the glass in upside down, it is because we are trying to support a dense fluid (the water) by one that is considerably less dense (the air). This is an *unstable* configuration. The smallest perturbation to the water surface becomes exponentially magnified with time, and a new equilibrium is then produced: water on the floor, an equilibrium solution no doubt well-known to the reader.

This process is an example of a *fluid instability.* Even an exact solution to the equations of motion may never be realized in nature. The solution must also be stable in the sense that small disturbances to the fluid remain small. The instability that we have been discussing, in which a light fluid supports a heavier one, is called the *Rayleigh-Taylor* instability. It is one of several classical fluid instabilities that have important applications and have been intensively studied in the laboratory and in numerical simulations. In astrophysics, the shock wave produced by a supernova can be decelerated by a much more rarified gas in the interstellar medium. The deceleration is equivalent to a gravitational field pointing from the denser to the less dense gas. This leads to a Rayleigh-Taylor instability, and the shock wave loses its coherence and can become turbulent.

9.2   Rayleigh-Taylor Instability

Consider a fluid of density ρ_2 on top of a fluid of density ρ_1. There is a gravitational field g present pointing downward from region 2 to region 1. At the interface $z = 0$ between the two regions, the surface is rippled with a vertical displacement ξ satisfying

$$\xi \propto \exp(ikx) \tag{626}$$

where k is a real constant. What is the subsequent development of the interface?

This problem is very analogous to the problem of surface water waves, and in fact, the basic equations applied separately in fluid 1 and fluid 2 are identical to the water wave problem. Only the surface interface boundary condition differs.

Let us assume a time dependence $\exp(-i\omega t)$ in all variables. The x dependence will follow that of the interface, $\exp(ikx)$, and from section (4.2) we know that the z dependence will be of the form $\exp(-|kz|)$ where the sign is determined by demanding regular behavior at large $|z|$.

The equations of motion in each of the fluids are

$$-i\omega \delta v_x = -ik\frac{\delta P}{\rho} \tag{627}$$

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$$-i\omega\delta v_z = -\frac{1}{\rho}\frac{\partial(\delta P)}{\partial z} \tag{628}$$

The boundary conditions at large $|z|$ imply that the fluid variables have a dependence of $\exp(|k|z)$ for the bottom fluid (1), and a dependence of $\exp(-|k|z)$ for the top fluid (2). The equations of motion then assure us that $\boldsymbol{\nabla \cdot \delta v} = 0$.

At the interface we have two boundary conditions. First, the fluid displacements (and in this case, the velocities) must be continuous at $z = 0$. Second, the force exerted on fluid 1 by fluid 2 must be equal to the force exerted by fluid 1 on fluid 2. In the absence of surface tension, this means that the pressures of the displaced fluid elements must be the same; the presence of surface tensions adds a restoring force of the form $-T\partial^2\xi/\partial x^2$. (See section 4.21 on capillary waves.) The force equation at the interface is then

$$\Delta P_1 = \Delta P_2 - T\frac{\partial^2\xi}{\partial x^2} \tag{629}$$

where the Lagrangian perturbation ΔP is defined by

$$\Delta P = \delta P + \xi\frac{\partial P}{\partial z} \tag{630}$$

(Can you give a physical argument why we need to use the Lagrangian perturbation and not the Eulerian?) Since the perturbed vertical velocity $\delta v_z = -i\omega\xi$ is continuous at $z = 0$, the z equations of motion for regions 1 and 2 state that

$$-i\omega\delta v_z = -\frac{|k|}{\rho_1}\delta P_1 \tag{631}$$

$$-i\omega\delta v_z = \frac{|k|}{\rho_2}\delta P_2 \tag{632}$$

The equilibrium pressure gradient is

$$\partial P_i/\partial z = -\rho_i g,$$

where i refers to either 1 or 2. (The equilibrium pressure at the interface is continuous at the interface, though its gradient is not.) Putting everything together, equation (629) becomes

$$\frac{i\omega}{|k|}\rho_1\,\delta v_z - i\frac{\delta v_z}{\omega}\rho_1 g = -\frac{i\omega}{|k|}\rho_2\,\delta v_z - i\frac{\delta v_z}{\omega}\rho_2 g + Tk^2\frac{i\delta v_z}{\omega} \tag{633}$$

which leads to

$$\omega^2(\rho_1 + \rho_2) = g|k|(\rho_1 - \rho_2) + T|k|k^2 \tag{634}$$

This is our dispersion relation. When $\rho_1 > \rho_2$, these disturbances propagate as waves. (These low frequency waves can be excited by slow-moving boats coming into a port, when a nearby river produces less dense fresh water lying on top of denser salt water. The waves are an important source of energy loss for the boat.) But if $\rho_1 < \rho_2$, wavenumbers less than

$$k_{crit}^2 = (\rho_2 - \rho_1)g/T \tag{635}$$

produce a negative ω^2. This means exponential behavior in time, and small disturbances grow explosively. This is the classic behavior of a fluid instability: small departures from equilibrium grow exponentially.

The underlying cause of the Rayleigh-Instability is that a heavy fluid lying on top of a less dense fluid is energetically unfavorable. The same system with the heavy fluid on the bottom is a state of lower (potential) energy. If a path to the lower state is opened, the system will exploit it. In our example the ripples between the two fluids create such a path, and they grow into long fingers of upwelling low density fluid and downwelling high density fluid, allowing the system to reach an equilibrium of lower energy.

9.3 The Kelvin-Helmholtz Instability

9.3.1 Simple homogeneous fluid.

Consider a fluid which is at rest in the half space $z < 0$, and moves with a velocity U in the $+x$ direction for $z > 0$. This is certainly a possible equilibrium state, but it is, in fact, unstable.

Denote the fluid at rest by subscript 1, and the fluid in motion by subscript 2. The density ρ is constant. The interface between fluids 1 and 2 is rippled with an assumed space-time dependence of $\exp(ikx - i\omega t)$. In fluid 1, the equation of motion is

$$-i\omega \boldsymbol{\delta v_1} = -\frac{1}{\rho}\boldsymbol{\nabla}\delta P_1 \tag{636}$$

while in fluid 2 it is

$$-i(\omega - kU)\boldsymbol{\delta v_2} = -\frac{1}{\rho}\boldsymbol{\nabla}\delta P_2 \tag{637}$$

With $\boldsymbol{\nabla}\cdot\boldsymbol{\delta v} = 0$, we see that δP in both fluids satisfies $\nabla^2\delta P = 0$, so that the z dependence of all variables is $\exp(-|kz|)$, as before. In particular, the

z equations of motion are

$$-i\omega \delta v_{z1} = -\frac{|k|}{\rho}\delta P_1 \tag{638}$$

$$-i(\omega - kU)\delta v_{z2} = \frac{|k|}{\rho}\delta P_2 \tag{639}$$

The *displacement* of the interface ξ must obviously be the same viewed from region 1 or 2, but not so the perturbed velocities δv_z. Indeed, in region 1,

$$\delta v_{z1} = \Delta v_{z1} = \frac{D\xi}{Dt} = -i\omega\xi \tag{640}$$

where the equality between Lagrangian and Eulerian perturbations follows because there are no velocity gradients in the background flow. In region 2, on the other hand,

$$\delta v_{z2} = \frac{D\xi}{Dt} = -i(\omega - kU)\xi \tag{641}$$

which is $\delta v_{z1} + ikU\xi$!

The condition of pressure balance in this case is entirely Eulerian

$$\delta P_1 = \delta P_2, \tag{642}$$

since there are no pressure gradients in the unperturbed equilibrium. Combining the last four equations gives a very unusual dispersion relation:

$$\omega^2 + (\omega - kU)^2 = 0. \tag{643}$$

Obviously this cannot be satisfied by any real value of ω! The solution is

$$2\omega = kU(1 \pm i) \tag{644}$$

and there is always an exponentially growing branch for any finite U.

This is the *Kelvin-Helmholtz* instability in its simplest form: two fluids in relative shear motion tend to be unstable. The source of free energy is obviously the shear itself. But what is the actual mechanism, why should shear be unstable? The answer can be found by a Bernoulli argument. An upward directed distortion of the interface into the upper region 2 causes a slight constriction for the x directed velocity. The fluid moves a little faster, to conserve mass. When it moves a little faster, the pressure drops, in accordance with Bernoulli's law. In region 1 underneath, however, the distortion causes a dilation in the flow, and the flow slows down. Thus the pressure rises from below, and as we have just argued, drops from above. The upward displacement is driven yet farther upward, and an instability ensues.

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9.3.2 Effects of gravity and surface tension

If *both* density stratification and surface tension are present, the Kelvin-Helmoltz instability can be suppressed. Short wavelengths are stabilized by surface tension, long wavelengths by rapidly propagating gravity waves. At sufficiently small velocities U, there are no wavelengths in between these limits that are unstable. Let us examine this problem in detail.

The system is a combination of the Rayleigh-Taylor and simple Kelvin-Helmholtz problems. There are two fluids as usual, number 1 below and number 2 above. Fluid 1 is at rest, fluid 2 moves at velocity U in the x direction, relative to it. As in the Rayleigh-Taylor problem, there are two densities: ρ_1 below and ρ_2 above. Here, we shall assume $\rho_1 > \rho_2$.

The space-time dependence of all fluid quantities is as in our previous two problems:

$$\delta X \propto \exp[i(kx - \omega t) - |kz|] \tag{645}$$

The vertical equations of motion is each fluid are

$$-i\omega \delta v_{z1} = -\frac{|k|}{\rho_1}\delta P_1 \tag{646}$$

$$-i(\omega - kU)\delta v_{z2} = -\frac{|k|}{\rho_2}\delta P_2 \tag{647}$$

The relationships between the δv_z's and ξ is given by equations (640) and (641) above. Hence

$$\delta P_1 = \frac{\rho_1 \omega^2}{|k|}\xi, \quad \delta P_2 = -(\omega - kU)^2\frac{\rho_2}{|k|}\xi \tag{648}$$

Finally, the condition of pressure continuity is exactly that of the Raleigh-Taylor problem, equation (629). Using this equation just as in RT problem, but with the above values of δP_1 and δP_2, we obtain the dispersion relation

$$\omega^2(\rho_1 + \rho_2) - 2\omega\rho kU + \rho_2 k^2 U^2 - |k|[g(\rho_1 - \rho_2) + k^2 T] = 0 \tag{649}$$

a combination of the Rayleigh-Taylor and simple Kelvin-Helmholtz formulae. We leave it as an exercise for the reader to show that this quadractic equation for ω has no instabilities if

$$\frac{\rho_1 \rho_2 U^2}{2(\rho_1 + \rho_2)} < \sqrt{gT(\rho_1 - \rho_2)} \tag{650}$$

Both gravity and surface tension are required for stabilization, as noted in the opening paragraph of this section. Even then, if the densities are close in value, the relative velocity is constrained very closely.

This behavior, in which both very long and very short wavelengths are stabilized, and instability is present only if there is a midrange of wavelengths that remains unstable, appears in a number of different problems. A classic example from astrophysics is gravitational instability in a rotating disk. Here, long wavelengths are stabilized by Coriolis forces (i.e., rotation), small wavelengths by pressure, and the disk is unstable if self-gravity is stronger than a sort of geometrical mean of the Coriolis and pressure forces. (See the text of Binney and Tremaine, *Galactic Dynamics,* for a detailed discussion.)

9.4   Stability of Continuous Shear Flow

The discussion of the Kelvin-Hemlholtz instability focused on a flow with a discontinuity in the velocity shear profile. The question naturally arises of whether instability occurs when the velocity changes continuously. Lord Rayleigh showed that for an inviscid flow, there is a very simple necessary, though not sufficient, condition: the velocity profile must contain an point of inflection at which its second derivative vanishes. This is known as the *Rayleigh inflection point criterion.* The argument is extremely clever.

9.4.1   Analysis of inflection point criterion

Consider a constant density velocity flow in the xy plane,

$$v_x = V(y)\boldsymbol{e_x} \tag{651}$$

We consider the behavior of small perturbations to this flow which depend upon x and t as $\exp[i(kx - \omega t)]$, with an amplitude that depends upon y. The linearized equations of motion are ($' \equiv d/dy$):

$$ik\delta v_x + \delta v_y' = 0 \tag{652}$$

$$-i(\omega - kV)\delta v_x + \delta v_y V' = -ik\delta P/\rho \tag{653}$$

$$-i(\omega - kV)\delta v_y = -\delta P'/\rho \tag{654}$$

These three equations can be reduced to a single equation for δv_y:

$$\delta v_y'' + \delta v_y \left(-k^2 + \frac{kV''}{\omega - kV}\right) = 0 \tag{655}$$

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Rayleigh's argument now proceeds as follows. Multiply the above differential equation by the complex conjugate δv_y^*, and integrate between upper and lower boundaries $\pm L$. Note that

$$\int_{-L}^{L} \delta v_y^* \, \delta v_y'' \, dy = \left[\delta v_y^* \, \delta v_y' \right]_{-L}^{L} - \int_{-L}^{L} |\delta v_y'|^2 \, dy \tag{656}$$

If either δv_y or its derivative vanishes at the boundaries, or if the boundary conditions are periodic, the integrated part vanishes. Under any of these conditions, the result of transforming the differential equation is

$$-\int_{-L}^{L} |\delta v_y'|^2 \, dy + \int_{-L}^{L} \left(-k^2 + \frac{kV''}{\omega - kV} \right) |\delta v_y^2| \, dy = 0. \tag{657}$$

If there is an instability present, then ω must have an imaginary part, ω_I. Writing $\omega = \omega_R + i\omega_I$, the imaginary part of this equation is

$$\omega_I k \int_{-L}^{L} \left(\frac{|\delta v_y^2| V''}{|\omega - kV|^2} \right) dy = 0. \tag{658}$$

A necessary condition for this equation to be satisfied is that V'' must be positive over part of the range of integration, and negative over other parts. In other words, it must pass through *zero*. The flow must have a point of inflection at which $V'' = 0$.

9.4.2 Viscous Theory

Viscous theory is considerably more complex, and we shall present only a summary of the results. The definitive numerical treatment of planar Poiseuille flow (viscous flow between plates at $z = \pm L$, $v_x \propto (L^2 - z^2)$) was accomplished in 1971. Instability is present only at Reynolds numbers $Re > 5772$. Note that there is no difficulty in duplicating the viscous velocity profile in the inviscid limit, since the inviscid shear profile $v_x(y)$ is completely unconstrained by the Euler equations of motion. Perturbations to the velocity, on the other hand, vanish at the boundaries for viscous flow but not for inviscid flow. In this sense there is not strict continuity between the two problems.

The role of viscosity is stabilizing both when the viscosity is large and when the viscosity is small; at intermediate values it destabilizes. The small Re limit is readily understood since the viscosity is strongly dissipative when it is large. High Re tends to be stabilizing because the inviscid flow is stable by the Rayleigh inflection point theorem. As seen in figure 16, however, high
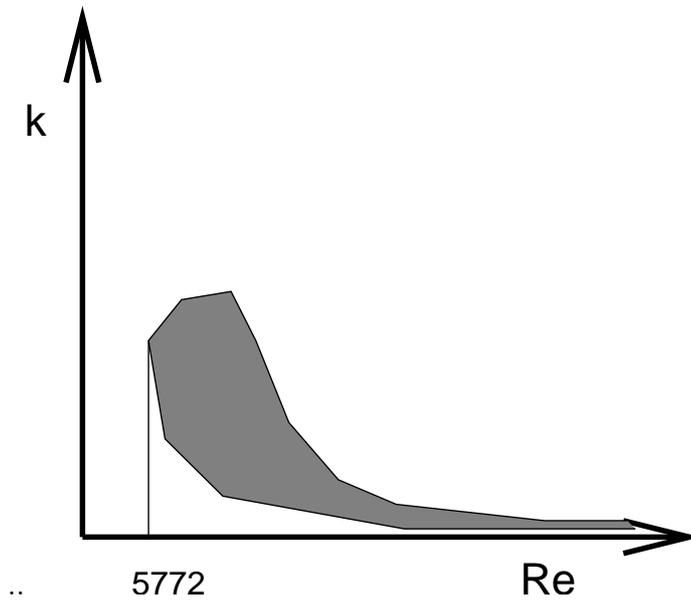
Figure 16: The shaded region is the zone of instability in the wavenumber–Reynolds number plane for classical Poiseuille flow. In the inviscid limit, this flow profile would be stable. Viscous flow theory always gives a zone of instability for any Reynolds number, though the width of the band goes to zero.

Re is never completely stabilizing: there are always unstable wavenumbers for any Re, but a smaller and smaller range as $Re \to \infty$. You can see how delicate the question of stability is for this problem! Viscosity is at once a stabilizing and destabilizing agent. By way of contrast, unstable flows *with inflection points* offer no such subtlety. At large Re there is an extensive range of wavenumbers that are always unstable.

9.5   Entropy and Angular Momentum Stratification

We end our very brief introduction to the vast topic of fluid stability by mentioning two additional classical instabilities. These are convective and rotational instability.

9.5.1 Convective instability

Convective instability has already been presented in our discussion of internal waves. These are waves that propagate in a medium that is stratified in the vertical z direction by an entropy gradient. We found that such waves obeyed the dispersion relation

$$\omega^2 = \frac{k_x^2}{k^2} N^2 \qquad (659)$$

where k_x is a wavenumber in the horizontal direction, $k^2 = k_x^2 + k_z^2$, and

$$N^2 = -\frac{1}{\gamma\rho}\frac{dP}{dz}\frac{d\ln P\rho^{-\gamma}}{dz} \qquad (660)$$

This may also be written in the form

$$N^2 = -g\frac{d\ln\rho}{dz} - \frac{g^2}{a^2} \qquad (661)$$

where a^2 is the adiabatic sound speed. In this form, the equation holds for both ideal gases as well as liquids. If

$$-\frac{d\ln\rho}{dz} < \frac{g^2}{a^2} \qquad (662)$$

then $N^2 < 0$, and small disturbances grow exponentially. Low density regions of the fluid are convected upward, high density regions downward. This generally corresponds to a net upward transport of heat. In the laboratory, heating a fluid from below lowers the magnitude of the density gradient, and induces this convective instability. It is generally necessary to include the effects of viscosity and thermal conduction in laboratory studies. The full problem is known as the *Rayleigh-Bénard* problem.

9.5.2 Rotational Instability

Consider a fluid in rotation about the z axis. The radial and azimuthal equations of motion for a constant density fluid are

$$\frac{\partial v_R}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_R - \frac{v_\phi^2}{R} = -\frac{1}{\rho}\frac{\partial P}{\partial R} - \frac{\partial\Phi}{\partial R} \qquad (663)$$

$$\frac{\partial v_\phi}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\phi + \frac{v_\phi v_R}{R} = -\frac{1}{R\rho}\frac{\partial P}{\partial \phi} - \frac{\partial\Phi}{R\partial\phi} \qquad (664)$$

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where we have allowed for the possibility of a central potential Φ. We consider motions in the $R\phi$ plane and small disturbances of the form $\exp(ikz - i\omega t)$. The linearized perturbation equations are

$$-i\omega\delta v_R - 2\Omega\delta v_\phi = 0 \tag{665}$$

$$-i\omega\delta v_\phi + \frac{\kappa^2}{2\Omega}\delta v_R = 0 \tag{666}$$

where $\Omega = v_\phi/R$ and

$$\kappa^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R} = \frac{1}{R^3}\frac{d(R^4\Omega^2)}{dR}. \tag{667}$$

κ is known as the epicyclic frequency in astrophysics, and κ^2 as Rayleigh's discriminant in the fluid community.

Once again, we look for solutions of the form $\exp(i\omega t)$. Such solutions exist, provided that

$$\omega^2 = \kappa^2 \tag{668}$$

Solutions are stable if $\kappa^2 > 0$, and unstable otherwise. In other words, *a rotating fluid is stable to axisymmetric disturbances if the angular momentum per unit mass $R^2\Omega$ increases outward.* This is known as the *Rayleigh criterion.* If we allow for more general axisymmetric wavenumbers $\boldsymbol{k} = (k_R, k_z)$, it is not difficult to show that the dispersion relation is

$$\omega^2 = \frac{k_z^2}{k^2}\kappa^2 \tag{669}$$

These disturbances propagate with a finite group velocity, and are known as inertial waves when $\kappa^2 > 0$.

Nonaxisymmetric disturbances are considerably more difficult to analyze (because of the presence of shear), and to this day a general stability criterion valid for both axisymmetric and nonaxisymmetric disturbances is still not known. It appears however, that the Rayleigh criterion is a good rule of thumb, except when the shear $d\Omega/d\ln R$ is larger than the Coriolis force 2Ω. Then the flow is prone to Kelvin-Helmholtz instabilities, because the Coriolis stabilization is too weak to compete with the destabilizing shear. Finally, a finite viscosity tends to stabilize the flow as well, and it is a classic and complex problem to determine the stability criterion that was first solved by G. I. Taylor in 1923. It is accordingly known as the Taylor problem.

In the laboratory, rotational stability is studied in what are known as Couette cylinders. An inner cylinder and an outer cylindrical shell enclose a

viscous liquid. By adjusting the rotation rates of the inner and outer cylinders, different interior flow profiles may be set up [see equations (475) and (476)]. Taylor not only derived the viscous stability criterion, but performed the difficult and painstaking experiment needed to confirm his result.

The parameter space for this problem is the plane defined by two axes: the Reynolds number for each of the cylinders, $R^2\Omega/\nu$, where R and Ω are the radius and rotation rate for each cylinder. (A recent experiment at Princeton University by Hantao Ji and his collaborators has achieved Reynolds numbers in excess of 10^6.) When the flow is unstable, an enormous variety of different types of different turbulent flows are seen in different regions of the parameter space, including one regime in which stable and turbulent flow are entwined with one another!

This completes our very brief introduction to the topic of fluid instabilities. We turn next to the study of turbulent flow, which is the general nonlinear outcome of the linear instabilities we have analyzed.

10   Turbulence

When the fluid instabilities of the previous section are allowed to develop fully into the nonlinear regime, the flow is characterised by extremely irregular spatial and temporal behavior. Often there is structure seen on all possible length and time scales, from global to microscopic. This is what is generally meant by *turbulence.* A more precise definition is not possible, because the phenomenon is too complex.

There is no theory of turbulence which is at a level, say, comparable to that of statistical mechanics. It is possible to be quite precise in calculating mean values of molecules and their statistical fluctuations in a gas in thermal equilibrium. It is not possible, however, to predict *a priori* what the mean RMS values of the velocity fluctuations will be in a turbulent flow, or how two components of the velocity fluctuations will be correlated with one another, or what the thermal energy transport will be due to the correlation between velocity and temperature fluctuations, or what the turbulent drag force on a moving body will be. For both enormously practical and purely intellectual reasons, we would love to be able to do this. It has been remarked that turbulence is a grand problem on the frontier of physics whose solution would have a tangible impact on our daily lives.

We are not wholly ignorant of all aspects of turbulence. The foundations of classical turbulence theory were established in the 1930's and 1940's by the Russian school of theoretical physics. Many important and semi-quantitative results were obtained by simple but very ingenious reasoning. We will discuss

some of these in the next section.

A radical shift in perspective occurred in the 1960's and 1970's. An influential work in 1963 by Lorenz showed that even simple systems of coupled differential equations display what we would now call *chaos.* Two well-behaved nonsingular solutions, separated by an infintesimal difference in their initial conditions, were found to completely diverge from one another after a relatively short time. This has very important implications in its own right (making, for example, long range weather forecasting impossible), but it also seemed to capture the essence of turbulence in a tractable mathematical setting. There was great excitement that real progress could be made. This excitement was heightened when experiments (including a real beauty by Libchaber here at the ENS) showed that crucial features of the mathematics, previously unobserved, were in fact seen directly in nature.

The intervening decades have not, alas, handed over the secrets of turbulence. Rather, they have taught us the right questions to ask, which is no small present. We now know that the problem is even more complex than imagined in the early days, and almost certainly no universal theory will be possible. But new mathematics, new physics, and a far deeper understanding of how the world works have all emerged from the efforts. Even if the problem is never fully solved, the struggle to understand the properties of turbulent flows will continue to be an enormously productive endeavor.

10.1   Classical Turbulence Theory

According to the Russian school, the onset of turbulence generally begins when the Reynolds $\mathcal{R}$ number exceeds some critical value. Large scale "eddies"—regions over which the velocity varies appreciably—appear first. At larger values of $\mathcal{R}$, eddies appear on smaller and smaller scales, and eventually they are present on all scales. The nonlinear interaction between two eddies produces structure on smaller scales, in much the same way that

$$\cos(k_1 x) \cos(k_2 x) = \frac{1}{2} \left[\cos(k_1 x + k_2 x) + \cos(k_1 x - k_2 x) \right] \qquad (670)$$

produces structure at the wavenumber $k_1 + k_2$ from structure at the smaller scales k_1 and k_2. (We take the wavenumbers to be positive.)

At the smallest scales at which eddies are present, λ_0, viscosity is important and the kinetic energy of the fluid turbulence is thermalized. This loss of mechanical energy at small scales must be replaced from sources at larger scales, which is just the role of the free energy source driving the instability (velocity shear, thermal gradients, etc.). We thus arrive at a picture of an energy cascade: power is injected at large scales, nonlinear interactions

bring energy to progressively smaller scales, and finally viscous dissipation thermalizes the energy at the smallest scales.

Turbulence is a highly dissipative process. The key idea of the Russian school is that a single quantity ϵ, the energy dissipated per unit time per unit mass of fluid, is the sole external physical parameter that determines the cascade of energy. It has dimensions of length^2 divided by time^3. A great deal may be deduced just by dimensional analysis alone.

10.1.1 Homogeneous, isotropic turbulence

On length scales well below the global scale of the problem, but well above the dissipation scale, we expect the properties of the turbulence to be homogeneous and isotropic. If one looks at velocity variations over a scale λ, how does this velocity v_λ depend upon λ? There is only one velocity that may be formed from ϵ and λ,

$$v_\lambda \sim (\epsilon\lambda)^{1/3} \tag{671}$$

What does this really tell us? We interpret this equation as follows. Pick a point in the fluid. Measure the velocity, v_1. At the same time, measure the velocity v_2 a distance λ away. Calculate the mean squared difference $(v_1 - v_2)^2$, and do this many, many times. That average variation calculated this way will be proportional to $\lambda^{2/3}$.

Similarly, if we follow a fluid element through its turbulent course over a time scale t, how large a velocity fluctuation are we likely to encounter? Once again, dimensional analysis suggests that only ϵ and t determine this velocity, hence

$$v_t \sim (\epsilon t)^{1/2} \tag{672}$$

The time variations experienced by a fluid element are proportional to the square root of time.

Be careful, however. If we fix our gaze at a single location of the fluid and ask how the mean velocity fluctuation behaves with time, we get a different result! How can that be? Dimensional analysis is supposed to provide a unique answer. The resolution of this apparent paradox is that at a fixed location, we are sensitive to the large scale velocity of the largest eddies (circulation patterns), Δu, which is *another* dimensional parameter. In time t, the fluid element passing in front of our eyes would have come from a distance $\lambda = (\Delta u)t$, and our result for v_λ then applies:

$$v_t(fixed\ location) = v_{\lambda(t)} = (\epsilon\,\Delta u\,t)^{1/3} \tag{673}$$

Notice that since we are concerned with very small times, this fixed location fluctuation v_t is larger than the intrinsic variation v_t (eq. [672]) of the in-

dividual fluid elements passing by the measurement device. Therefore the fluctuations represented by equation (673) are more important than the time fluctuations of a single fluid element. If the fluctuations of the individual fluid element had been larger than those caused by the spatial variation represented in v_λ, then the latter would have been unimportant, and our fixed location and fixed fluid element results would have been the same: both would be given by the fixed fluid element result. (You begin to get a sense for the subtleties of describing turbulent quantities!)

The spatial variations represented in v_λ imply that there is an associated Fourier power spectrum, $v^2(k)$. The dimensions of $v^2(k)$ are velocity^2 per unit wavenumber, or length^3 per time^2. On scales well below global and well above dissipative, the power spectrum can depend only upon ϵ and, of course, k. (Only the magnitude of k enters, since we assume the flow is isotropic.) Dimensional analysis then gives

$$v^2(k) \sim \epsilon^{2/3} k^{-5/3} \tag{674}$$

This is the celebrated energy power spectrum for homogeneous isotropic turbulence first derived by Kolmogorov in 1941. It has been seen in the laboratory, in numerical simulatons, and in nature. Notice that most of the power in concentrated at the largest scales (small k).

It is useful to have temporal power spectra as well, especially for laboratory measurements. Once again, we have to be careful! The energy per mode quite generally scales as $(\epsilon/k)^{2/3}$. The question is which frequency should be associated with a given wavenumber k? This frequency can always be written $\omega = kc$, where c is some velocity. It is this velocity that differs between Eulerian (spatially fixed) and Lagrangian (fluid element fixed) spectra. For the Eulerian power spectrum, c is Δu, as before, the dominant large scale velocity drifting across our observation point. Hence the energy per mode per unit frequency is

$$v^2(\omega) \text{ [Eulerian]} \sim (\epsilon/k)^{2/3} \omega^{-1} \sim (\epsilon \Delta u)^{2/3} \omega^{-5/3}. \tag{675}$$

For the Lagrangian fluid element, only local quantites enter, which means $c \sim (\epsilon/k)^{1/3}$ and

$$\omega \sim k(\epsilon/k)^{1/3} = \epsilon^{1/3} k^{2/3} \tag{676}$$

In this case,

$$v^2(\omega) \text{ [Lagrangian]} \sim (\epsilon/k)^{2/3} \omega^{-1} \sim \epsilon \omega^{-2}. \tag{677}$$

In a typical laboratory arrangement, a very fine wire is inserted in the flow, and velocity fluctuations of turbulent gas passing over the wire result in cooling by the "wind". This, in turn, induces fluctuations of the wire temperature. The ohmic resistance of the wire is very temperature sensitive

(by design), and therefore the electrical current fluctuations measure the velocity fluctuations in the flow–one hopes that the hot wire doesn't itself affect the flow it is trying to measure! The expected temporal spectrum for this set-up is Eulerian, $\omega^{-5/3}$, and this has indeed been well-confirmed.

Finally, let us consider the effects of viscosity. The most important question is at what scale in the energy cascade do we expect viscosity to alter the $-5/3$ law? A first glance, one might think that it is only at the scale of the particle mean free path that we expect viscous effects to be important, as we found when we studied shock waves. But in fact, viscous effects enter at much larger scales, essentially because the characteristic length scales associated with ϵ are large. There is only one quantity that can be formed from ϵ and ν that has the dimensions of length:

$$\lambda_0 \sim (\nu^3/\epsilon)^{1/4} \tag{678}$$

This may also be derived by setting

$$\epsilon \sim \nu(\epsilon\lambda_0^{1/3})^2 \; \lambda_0^{-2}, \tag{679}$$

which balances the energy dissipation rate in the cascade by viscous losses. This show the characteristic velocity that is being directly dissipated: $(\epsilon\lambda_0)^{1/3}$. If we now express ϵ in terms of the large scale eddy velocity Δu and length l, then with $\epsilon \sim (\Delta u)^3/l$,

$$\lambda_0 \sim l \; \mathcal{R}^{-3/4} \tag{680}$$

where the large scale Reynolds number is $\mathcal{R} \sim l\Delta u/\nu$. The viscous scale of the turbulence is proportional to the $-3/4$ power of $\mathcal{R}$.

10.2 The decay of free turbulence

This section contains optional advanced material, included for your interest.

A classical result shown by Kolmogorv in 1941 is that free, undriven turbulence will decay with time, with the energy density following a $t^{-10/7}$ power law. If, as the generally the case, the region of turbulence is bounded by quiescent fluid, the size of the turbulent region will actually *increase* in time, proportional to $t^{2/7}$. Kolomogorov's argument was rather complicated, but Landau & Lifschitz showed how these scalings arise from global angular momentum conservation of the fluid. We follow their line of argument.

The kth component of angular momentum of a constant density fluid may be written

$$M_{ij} = \rho \int (x_i v_j - x_j v_i) \, dV \tag{681}$$

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where the integral is taken over the volume of the fluid, of order L^3, say. By contrast, the largest coherent scale of the turbulence is taken to be l, $l \ll L$. Turbulence is present at scales large compared with l, but small compared with L. The identity

$$\frac{\partial(x_i x_j v_l)}{\partial x_l} = x_i v_j + x_j v_i,$$ (682)

valid for a divergence free velocity v_l may be used to show

$$\int (x_i v_j + x_j v_i)\, dV = 0,$$ (683)

provided that the normal velocity vanishes at the boundary of the fluid. (We are of course using the divergence theorem on the left side of equation [682].) Hence

$$M_{ij} = 2\rho \int x_i v_j\, dV$$ (684)

For $i = x$ and $j = y$, M_{xy} is the z component of the angular momentum, and M_{yx} is the $-z$ component. The same holds for other permutations of and i and j, and if $i = j$ then M_{ij} vanishes. Therefore

$$M_{ij}M_{ij} = 4\rho^2 \left(\int x_i v_j\, dV \int x_i' v_j'\, dV' \right) = \text{constant}$$ (685)

since this is just twice the square of the conserved angular momentum vector. The repeated integral may be written as a double integral

$$\int x_i v_j\, dV \int x_i' v_j'\, dV' = \int \int x_i x_i' v_j v_j'\, dV\, dV'$$ (686)

and the double integral itself may be written as

$$I \equiv \int \int x_i x_i' v_j v_j'\, dV\, dV' = -\frac{1}{2} \int \int (x_i - x_i')^2 v_j v_j'\, dV\, dV'.$$ (687)

This last equality follows since the term involving $(x_i)^2$) or $(x_i')^2$ involve the direct integration of either v_j or v_j' by itself over the volume of the fluid. This, of course, must vanish.

The integral of interest becomes

$$I = -\frac{1}{2} \int dV \int (x_i - x_i')^2 v_j v_j'\, dV' \equiv -\frac{1}{2} \int dV \int s^2 \boldsymbol{v'} \boldsymbol{\cdot} \boldsymbol{v}\, dV'$$ (688)

where $s^2 = (x_i - x_i')^2$ is the distance between the two points x_i and x_i' in the fluid. We now perform the following rather subtle averaging procedure.

Fix s, and average the velocity dot product over all x_i and x_i'. Then do the integration. This is the same as peforming the average over the double integral itself. Because the turbulence is isotropic, the average

$$\langle \boldsymbol{v'} \!\cdot\! \boldsymbol{v} \rangle$$

must itself be a function only of s, let us say $F(s)$. We are in essence calculating the correlation of the velocity field in the turbulent fluid, and this correlation should fall rapidly to zero on scales larger than the largest coherent eddy, l. Therefore

$$G \equiv \int s^2 F(s) dV' \tag{689}$$

will depend upon l, but not x_i. (This is an important and subtle point, the key to the argument, and worth some thought.) We conclude that

$$I = -\frac{1}{2} \int dV \int s^2 F(s) dV' = -\frac{GV}{2}. \tag{690}$$

where V is the assumed large but finite volume of the fluid. But if I is constant in time (conserved angular momentum), then G *must be also be constant in time.* Since G scales like $v^2 l^5$, we conclude that

$$v^2 l^5 = \text{constant in time} \tag{691}$$

In this scale-free problem, we must have $v \sim lt$, and one finds Kolomogov laws for the decay of free, isotropic turbulence,

$$v^2 = \text{constant}/t^{10/7} \tag{692}$$

and

$$l = \text{constant} \times t^{2/7} \tag{693}$$

A beautiful and delicate argument. Too bad it is probably wrong in detail! The problem is related to the existence of G: this integral need not converge. (It is hard to converge with an r^4 factor in the integrand.) However, experiments are in basic, if somewhat approximate, agreement with this Kolmogorov decay formula, so most researchers feel that this is not all complete nonsense. Recently, this problem has been put on a more firm foundation by Yakot (2004, *J. Fluid Mech.,* **505,** 87), who claims to have recovered Kolmogorov scaling, but without demanding the existence of G. You can see that one does not have to go very far into turbulence theory before unresolved issues are present! The free decay of isotropic turbulence *still* remains to be fully solved.

10.3 Turbulence: a modern perspective

The story begins in 1963. The meteorologist E. A. Lorenz was interested in the results of certain thermal experiments conducted in a rotating ring, or annulus. The inner wall of the annulus was cooled and the outer wall was heated. Because of the outward centrifugal force, the sense of this heating was similar to heating a nonrotating system from below in a gravitational field. Both walls of the annulus rotated at the same angular frequency Ω. Despite the fact that there was no differential rotation in the confining walls, differential rotation was established in the fluid by the thermal convection that was produced by the instability. As Ω increased, amplified nonaxisymmetric waves that were present turned into a large scale stream (something like that atmospheric jet stream). This stream could be steady, oscillatory, or highly non-periodic depending upon the value of Ω.

The presence of large scale organization within a turbulent flow is a major omission of classical theory. A signature of what is now called *chaos* is to see both periodic and completely irregular behavior in the same system, with only a slightly different set of external parameters. A very simple example of an elementary equation that shows chaotic behavior is known as the logistic map, first studied in detail in 1976 by Robert May (now *Lord* May!) as a model for biological populations:

$$x_{n+1} = rx_n(1 - x_n) \qquad (694)$$

This is basically an equation for geometrical growth with the simplest possible nonlinear quenching. It works as follows: Choose a random starting point x_0 in $[0, 1]$. Compute x_1 from the equation, then use x_1 to compute x_2, and so on. This may be thought of an evolutionary equation for x. When $r < 4$, all of the x_n stay within $[0, 1]$. $x = 1 - 1/r$ is a fixed point, but the mapping actually converges to this value only if $r < 3$. For $3.449 > r > 3$, the convergence oscillates between two values, when $r > 3.499$ it oscillates between four values: there is a *period 4* solution. As r increases, period doubling continues indefinitely, *but it occurs at smaller and smaller increments of r.* The limiting value of this kind of period doubling sequence is $r = 3.570$. Above this number, the behavior is chaotic, with only occasional values of r once again giving rise to periodic behavior. (See figure 17.) At around 3.81, a period of three is seen. It is has been proven that any mapping that has a period of three must always also show chaotic behavior.

As $r \to 3.570$, we may identify a sequence r_m at which each new period doubling occurs. If we now evaluate the limit of

$$\frac{r_m - r_{m-1}}{r_{m+1} - r_m}, \quad \text{as } m \to \infty \qquad (695)$$

we find that it is equal to 4.66920160910299.... This is known as *Feignbaum's number.* M. Feigenbaum has proved that this is a universal behavior of

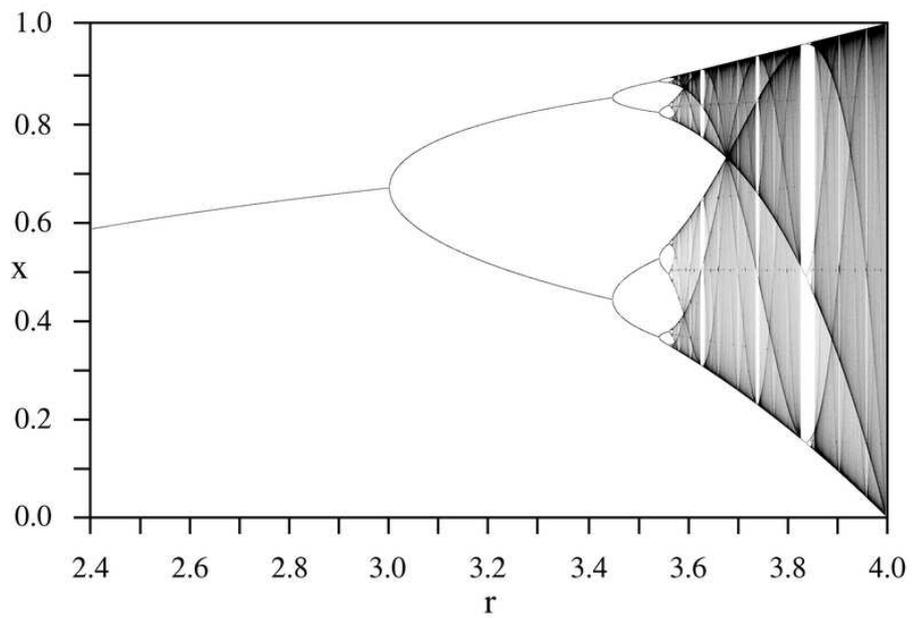Figure 17: Convergence points of the logistic equation as a function of r. For $r < 3.570$, discrete points exist, and convergence is periodic. For $r > 3.570$ the response is chaotic and a distributional response is shown instead.

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period doublings of any map of the form $x_{n+1} = f(x_n)$, no matter what the function f is! Feigenbaum's number is in this sense a fundamental universal mathematical constant, like e or π. In fact, Feigenbaum's theory gives even more than this (truly remarkable) result. It predicts, for example, the values of the external r parameter at which period doubling occurs, and even the relative strength of each period (i.e., the power at different frequencies).

Nice mathematics, but where is the physics? What do we use for our "maps?" The answer is that the maps of physics are partial differential equations, which are in a sense just elaborate prescriptions that tell us how to get new values of flow (or field) quantities from old values. In its most direct interpretation, a partial differential equation tells us to apply a differencing operator to a set of old data values in order to calculate the next series of values. Indeed, on a computer, this is quite literally what you do. So from the very beginning, there was always some hope that the results of Feigenbaum's functional theory might actually be seen in nature.

In 1977, Albert Libchaber of the ENS designed and carried out a remarkable experiment involving liquid helium in a tiny cell in which temperature changes of a few thousandths of a degree could induce convective instability.[7] By carefully increasing the heating, he found *exactly* the same march of period doublings seen in the logistic map, but in this case they led to the onset of turbulence! It was stunning confirmation that the period doubling route to turbulence was more than a mathematical possibility.

But the problem of turbulence has not been solved just yet. Turbulence also appears by routes other than period doubling. In Couette experiments, for example, vortices appear which have nonaxisymmetric structure. As the inner cylinder increases its rotation rate, this "wavy vortex" becomes unstable and generates more vortices, but with a period that is not a simple multiple of the first. At yet higher Ω, there is a sudden burst of broadband noise. This was in fact foreseen by Ruelle and Takens, who developed a detailed theory based on mapping techniques applied to coupled ODE's. They predicted that there would be this additional route to turbulence, in addition to progressive period doublings.

In some cases, turbulence is triggered *not* by the gradual adjustment of some global parameter, like the heating rate or the rotation rate. Instead it occurs in the normal course of evolution of an initially smooth fluid, say, as it traverses a confining wall. For example, fluid flowing over a flat plate will develop a boundary layer whose thickness grows in proportion to $\sqrt{x}$, where x is the horizontal distance from the leading edge of the plate (see section 9.1). The boundary layer in this case does not separate, since there are no adverse gradients causing it to peel away from the survace. Nevertheless,

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[7]See *Chaos*, by James Gleick, for a highly informative and entertaining discussion of this topic.

when the boundary layer has grown so that the internal Reynolds number $U\delta/\nu \sim 500$ (U is the flow velocity, δ the boundary layer thickness, ν the kinematic viscosity), turbulence starts to develop. The mechanism is similar to Poisseuille and planar Couette flow. At earlier points in the flow, the boundary layer is too thin and too viscous to be unstable, but as it spreads vertically, the Reynolds number drops below the critical value needed for the onset of instability.

In this example, the onset of turbulence goes through several stages. First two-dimensional waves appear, with wave crests perpendicular to the direction of the unperturbed velocity. These waves, rather than the boundary layer flow directly, then feed the turbulence. First, the waves develop their own instabilities, forming growing ripples along their wave crests. These get increasingly more irregular and agitated, breaking down into turbulent flow. But the turbulence occurs only in discrete, well defined regions! Finally, even further downstream, the turbulent regions merge, and a extended plane of fully developed turbulence is seen.

The problem of self-sustained (as opposed to externally driven) turbulence is truly three-dimensional. In shear flow, in which the ultimate source is the velocity gradient, the link between the fluctuations and the free energy source is provided by vortex stretching. Vortices are, as we know, frozen into the flow. When stretched by shearing motion, the vortex compensates by increasing the circulation around its core. This process effectively transfers large scale shear into smaller scale eddies. Ultimately, this energy is cascaded down and thermalized at the viscous scale. Because vorticies are responsible for binding turbulence to large scale shear, the fluid dynamicist Keith Moffatt has referred to them as the "sinews" (= tendons) of turbulence.

Better is the end of a thing than the beginning thereof...

— Ecclesiates 7:8