MAGNETOHYDRODYNAMICS Année 2010-2011

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Please do all questions, indicating your answers clearly in your books. All equations that you require are provided in the problems or at the end of the exam. Vous pouvez répondre soit en français, soit en anglais, mais je vous encourage à écrire en anglais, si c'est possible. There will be no penalty for incorrect English, as long it describes correct physics! Whichever language you choose, write clearly.

Feel free to ask if you do not understand what is being requested in a question. You may use a result from earlier in problem later in the problem, even if you were not able to derive it.

Strong magnetic fields in the interstellar medium:
the Parker instability

1.) In this exam, we will study what is known as the *Parker Instability,* an important process in the interstellar medium that disturbs strong magnetic fields when a gravitational field is present.

1A.) Show that if a displacement $\boldsymbol{\xi}$ is made to a fluid element, then the change in the magnetic field $\boldsymbol{\delta B}$ at the original location of the element is

$$\boldsymbol{\delta B} = \boldsymbol{\nabla}\times(\boldsymbol{\xi}\times\boldsymbol{B})$$

where $\boldsymbol{B}$ is the equilibrium magnetic field. As a "petit conseil", start with the induction equation and justify the statement that

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\boldsymbol{\delta B}}{\delta t}$$

1B.) Using exactly the same reasoning, show that the entropy equation for an adiabatic perturbation implies

$$\frac{\delta P}{P} - \gamma\frac{\delta\rho}{\rho} = -\boldsymbol{\xi}\boldsymbol{\cdot}\boldsymbol{\nabla}\ln P\rho^{-\gamma}$$

where P is the gas pressure, ρ is the density, and γ is the adiabatic index.

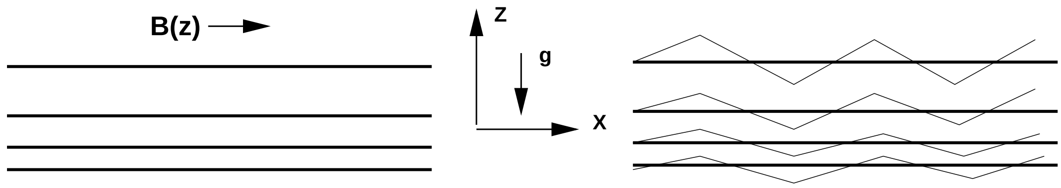

Figure 1: *Diagram for the Parker Instability. The equilibrium field (left) is horizontal and depends upon z only. The perturbed field (right) results from vertical fluid displacements of the form* $\exp(ikx - i\omega t)$. *There is a uniform gravitational field g pointing downward.*

1C.) Consider next a gas of uniform temperature in the presence of a constant z (vertical) gravitational field g. The horizontal direction is x. There is a magnetic field in the x direction, $B(z)\boldsymbol{e_x}$ that depends only on z. The ratio of gas pressure to magnetic pressure is

$$\beta \equiv \frac{2\mu_0 P}{B^2},$$

and it is a constant, independent of position and time, in the equilibrium state. Prove that ρ, P, and B^2 are all proportional to $\exp(-z/H)$ in the equilibrium state, where

$$H \equiv \frac{C_S^2}{g}\left(\frac{1+\beta}{\beta}\right)$$

is the scale height, and C_S is the constant isothermal sound speed $\sqrt{P/\rho}$.

We will use this solution as a very simple background model for the vertical structure of the Galaxy.

1D.) We next make a small disturbance to our equilibrium state by introducing a small vertical fluid displacement ξ. ξ is assumed to be proportional to

$$\exp(ikx - i\omega t),$$

as are the pertubed quantities $\delta\rho/\rho$, $\delta P/P$, and $\boldsymbol{\delta B}/B$. Notice, however, that since ρ, P and B all depend upon z, so must $\delta\rho$, δP, and $\boldsymbol{\delta B}$ also depend upon z.

Show that the linearly perturbed equation of motion is

$$-\omega^2\xi\boldsymbol{e_z} = -\frac{\boldsymbol{\nabla}}{\rho}\left(\delta P + \frac{B\delta B_x}{\mu_0}\right) - \frac{\delta\rho}{\rho}g\boldsymbol{e_z} + \frac{B\partial_x\boldsymbol{\delta B}}{\rho\mu_0} + \frac{\boldsymbol{\delta B}\boldsymbol{\cdot}\boldsymbol{\nabla}B}{\rho\mu_0}\boldsymbol{e_x}$$

and that

$$\boldsymbol{\delta B} = \boldsymbol{e_z} B\partial_x\xi - \boldsymbol{e_x}\xi\partial_z B$$

1E.) By solving the x component of the equation of motion, show that the pressure change δP is

$$\frac{\delta P}{\rho} = -\frac{C_S^2\xi}{\beta H}.$$

Thus, there is a lowering (raising) of the pressure for an upward (downward) displacement. There is an important distinction between the purely hydrodynamical problem (without a magnetic field), for which vertical displacements cause no pressure change, and the magnetic problem, which has such a pressure change.

1F.) By solving the z equation of motion, show that the displacement ξ obeys the very simple equation

$$-\omega^2\xi = -g\frac{\delta\rho}{\rho} - k^2 v_A^2\xi$$

where, as usual, the Alfvén velocity is $v_A^2 = B^2/\rho\mu_0$. Notice that only magnetic tension and buoyancy ("la force d'Archimède") are present.

1G.) Using the entropy equation to solve for the density perturbation, derive the dispersion relation for the Parker problem:

$$\omega^2 = \frac{\gamma-1}{\gamma}\frac{g}{H}\left(1 - \frac{1}{\beta(\gamma-1)}\right) + k^2 v_A^2$$

If $\beta < 1/(\gamma-1)$ the gas is unstable. Explain why.

1H.) The classical instability criterion for a nonmagnetized, vertically stratified gas is due to Karl Schwarzschild (of black hole fame). It is

$$\frac{d\ln P\rho^{-\gamma}}{dz} < 0 \qquad \text{(for instability)} \qquad [1]$$

which states that the entropy decreases upward for instability. Show that this entropy criterion may also be written as

$$\frac{\rho g}{\gamma P} + \frac{1}{\rho}\frac{d\rho}{dz} > 0 \qquad [2]$$

where g is the (nonmagnetized) equilibrium gravitational field. Now, show that if we include a horizontal magnetic field, equation [1] is not the correct criterion, but that equation [2] leads to $\beta < 1/(\gamma - 1)$, which *is* the correct criterion!

If all has gone well, you now understand an important MHD process in the interstellar medium. The "Parker Instability" (which was well-known to plasma physicists many years before Parker's work) does not allow a strong magnetic field to take the place of the usual gas pressure in supporting the interstellar medium against the pull of gravity. For a diatomic molecular gas, your results show that if β is less than 2.5, the medium is unstable, with denser gas sinking and less dense gas rising. In a classical early model of star formation, the dense gas that collects in magnetic U-shaped valleys was identified with sites of gravitational collapse and active star forming regions.

USEFUL EQUATIONS

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\frac{1}{\rho}\boldsymbol{\nabla}\left(P + \frac{B^2}{2\mu_0}\right) - \boldsymbol{\nabla}\Phi + \frac{1}{\mu_0\rho}(\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$$

$$\left[\frac{\partial}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\right]\ln P\rho^{-\gamma} = 0 \qquad \text{(adiabatic process)}$$

$$\epsilon^{ijk}\epsilon^{mlk} = \delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{A}\times\boldsymbol{B}) = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{A}\boldsymbol{\nabla}\cdot\boldsymbol{B} - \boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{A}$$