Mon. Not. R. Astron. Soc. **356,** 131–144 (2005) doi:10.1111/j.1365-2966.2004.08428.x

A two-stream formalism for the convective Urca process

P. Lesaffre,[1,2]★ Ph. Podsiadlowski[2] and C. A. Tout[1]

[1]*Institute of Astronomy, Madingley Road, Cambridge CB3 0HA*
[2]*Department of Astrophysics, University of Oxford, Oxford OX1 3RH*

Accepted 2004 September 22. Received 2004 September 21; in original form 2004 August 11

ABSTRACT

We derive a new formalism for convective motions involving two radial flows. This formalism provides a framework for convective models that guarantees consistency for the chemistry and the energy budget in the flows, allows time dependence and accounts for the interaction of the convective motions with the global contraction or expansion of the star. In the one-stream limit, the formalism reproduces several existing convective models and allows them to treat the chemistry in the flows. We suggest a version of the formalism that can be implemented easily in a stellar evolution code. We then apply the formalism to convective Urca cores in Chandrasekhar-mass white dwarfs and compare it to previous studies. We demonstrate that, in degenerate matter, nuclear reactions which change the number of electrons strongly influence the convective velocities, and we show that the net energy budget is sensitive to the mixing. We illustrate our model by computing stationary convective cores with Urca nuclei. Even a very small mass fraction of Urca nuclei (as little as 10^{-8}) strongly influences the convective velocities. We conclude that the proper modelling of the Urca process is essential for determining the ignition conditions for the thermonuclear runaway in Chandrasekhar-mass white dwarfs.

Key words: convection – neutrinos – nuclear reactions, nucleosynthesis, abundances – supernovae: general – white dwarfs.

1 INTRODUCTION

Gamow & Schönberg (1941) were the first to recognize the Urca process (electron captures and emissions on pairs of nuclei that can be converted into each other by electron captures/beta decays) as a potentially strong source of neutrino cooling in degenerate stars. This process is already responsible for significant cooling during the late radiative phase of accreting C/O white dwarfs. For each Urca pair, the cooling occurs at a mass shell, a so-called Urca shell, determined by the characteristic density for the pair at which the electron captures/beta decays take place. When carbon burning starts, a convective core grows and soon engulfs the Urca shells. The convective motions across the Urca shells back and forth directly affect the net energy release as well as the net amount of electrons captured. The resulting phenomenon is called the convective Urca process and is a key ingredient in linking the late evolution of the progenitor of a Type Ia supernova (SN Ia) with the subsequent explosion (Paczyński 1972).

Over the last 30 years, there have been numerous studies of the convective Urca process with often mutually exclusive conclusions. Bruenn (1973) realized that nuclear heating in Urca matter outside chemical equilibrium dominates over the neutrino losses. On the other hand, Couch & Arnett (1975) stressed the cooling effect of the work done by convection. In the most detailed work to date, Iben (1978a,b, 1982) computed the evolution of an accreting white dwarf including the detailed chemistry of many Urca pairs. He realized that the turn-over time-scales would be of the same order as the chemical time-scales for the Urca reactions. This, he concluded, implied that the mixing processes caused by the growth of the convective core would affect the heating/cooling by the Urca process. However, numerical problems caused by his treatment of the mixing prevented him from following the computations up to the thermal runaway.

Barkat & Wheeler (1990) revisited the problem of the convective Urca process and provided a clear summary of the convective Urca mechanism, although later Mochkovitch (1996) and Stein, Barkat & Wheeler (1999) pointed out a mistake in their treatment and argued that more attention needs to be paid to the kinetic energy of convection. Bisnovatyi-Kogan (2001) showed that the feedback of the Urca process on convection itself should be taken into account. As this summary shows, a consistent picture of convection that properly treats the chemistry is still missing and is badly needed in order to address the problem of the convective Urca process.

Such a theory for chemistry coupled with convection was attempted by Eggleton (1983) with the help of a 'simple rule-of-thumb procedure'. Later, Grossman, Narayan & Arnett (1993) produced a physically consistent model of convection, which includes chemistry based on a statistical approach for the convective blobs.

★E-mail: lesaffre@ast.cam.ac.uk

© 2004 RAS

However, their model did not necessarily conserve energy and has only been checked without chemistry.

Here we devise a model for convection that ensures energy conservation. We start with the conservation equations of radiation hydrodynamics in a spherical configuration. Using a simple geometry, which mimics convective rising plumes, we derive a physically self-consistent model of convection that includes time-dependent chemistry. We then compare this formalism to previously derived models of convection. In the process we obtain a model for the convective Urca process that addresses all the problems mentioned above: nuclear heating, mixing, convective work, kinetic energy and the feedback of the Urca process on convection. To illustrate the formalism, we apply it to stationary convective Urca regions and show how it affects the energy budget, the convective properties and the chemical stratification. In a follow-up paper, we plan to apply the formalism in a realistic, time-dependent stellar model, coupled with a complete nuclear reaction network, to determine the ignition conditions for the thermonuclear runaway in a SN Ia.

In Section 2 we derive the basic equations for the two-stream formalism and suggest a simple model for the exchange of matter, momentum and energy between rising and descending flows. In Section 3 we compare our model to existing models of convection. In Section 4 we describe the convective Urca process in view of our model. In Section 5 we compute stationary Urca convective cores. We discuss and summarize our results in Sections 6 and 7.

2 TWO-STREAM FORMALISM FOR CONVECTION

Cannon (1993) designed a two-stream algorithm to post-process the evolution of chemical species in convective regions. This model was well suited to the study of convective regions in which the chemical time-scales were shorter than the convective turn-over time-scales. However, as a post-processing algorithm, it did not tackle the feedback effects the chemistry could have on the convection. Here we extend his ideas to all state variables in the two streams and explore the interactions between mass, energy, momentum and chemical transfers between the streams.

Let us consider a sphere of gas with purely radial velocities and stratified properties. We assume that, on a shell of radius r, there are two different velocities associated with the two streams. We further assume that, on this shell, all gas parcels moving with a given velocity have the same state, and they have homogeneous temperature, pressure and chemical composition. Fig. 1 schematically shows the geometrical configuration we have in mind.

One of the two velocities has to be greater than the other, and we refer to the gas moving with this velocity as the upward moving gas, even though both velocities could be negative in principle. We write the two velocities as $v + v_\mathrm{u}$ and $v - v_\mathrm{d}$, where v_u and v_d are both positive and are associated with the upward and downward moving fluid, respectively. The ambient velocity v will be defined further below.

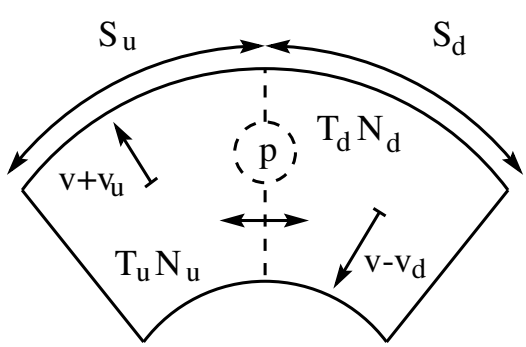

Figure 1. Geometrical configuration of the two streams.

We now affix suffixes 'u' to upward moving fluid properties and 'd' to downward moving fluid properties. We assume that the sound-crossing time of a horizontal section of one stream is short compared to any other time-scale. Hence, we make the approximation that the pressure $p = p_\mathrm{u} = p_\mathrm{d}$ is the same in the upward and downward moving fluids. The state of the gas in both fluids is now completely determined by the temperatures T_u and T_d and abundance vectors $\boldsymbol{N}_\mathrm{u}$ and $\boldsymbol{N}_\mathrm{d}$ (number of particles per unit mass), provided we know the equation of state of the gas. The latter equation provides the mass densities $\rho_\mathrm{u}(p, T_\mathrm{u}, \boldsymbol{N}_\mathrm{u})$ and $\rho_\mathrm{d}(p, T_\mathrm{d}, \boldsymbol{N}_\mathrm{d})$, and the specific energies e_u and e_d, along with the radiative volumic energies E_u and E_d.

We define S_u and S_d as the areas of the surfaces occupied by the two fluids at the shell of radius r. Therefore

$$S = S_\mathrm{u} + S_\mathrm{d} = 4\pi r^2 \tag{1}$$

and we define the velocity v by setting the mass flow

$$\dot{m} = S_\mathrm{u} v_\mathrm{u} \rho_\mathrm{u} = S_\mathrm{d} v_\mathrm{d} \rho_\mathrm{d}. \tag{2}$$

This defines v as the radial velocity of the centre of a mass shell. The net mass flow through the shell moving at velocity v is zero.

We now write the equations for the variation of mass, momentum and energy for each of the two fluids and define the exchange terms. We then compute the equations for the average fluid and the specific equations of evolution for both fluids, before computing the differential evolution between the two fluids. Finally, we propose a very simple model for the exchange terms between the streams.

2.1 Conservation equations

We consider the mass, momentum and energy on a shell at radius r for each of the two fluids. This allows us to define the exchange terms in a conservative way and makes it easier to derive the mean equations. Moreover, with this approach the specific exchange terms can be defined more rigorously.

The viscosity of the fluid is neglected as well as the molecular diffusion. All horizontal effects are implicitly included in the exchange terms.

2.1.1 Mass

To simplify the derivation, we neglect the mass changes due to the nuclear transformations and assume that mass is perfectly conserved. We treat the corresponding nuclear energy production only in the energy equation. The rate of change of mass is hence equal to the sum of the mass flux in the radial direction and sideways

$$\frac{\partial}{\partial t}(S_\mathrm{u}\rho_\mathrm{u}) = -\frac{\partial}{\partial r}[S_\mathrm{u}\rho_\mathrm{u}(v + v_\mathrm{u})] + \dot{M}, \tag{3}$$

and

$$\frac{\partial}{\partial t}(S_\mathrm{d}\rho_\mathrm{d}) = -\frac{\partial}{\partial r}[S_\mathrm{d}\rho_\mathrm{d}(v - v_\mathrm{d})] - \dot{M}, \tag{4}$$

where $\dot{M}$ represents the mass per unit radius and unit time exchanged in the shell from the downward to the upward moving fluid.

2.1.2 Momentum

The rate of change of momentum is the sum of the momentum flux (ram and thermal pressure) in the radial direction and sideways, added to the gravitational forces where the gravitational potential is assumed to be spherical:

$$\frac{\partial}{\partial t}[S_{\rm u}\rho_{\rm u}(v+v_{\rm u})] = -S_{\rm u}\frac{\partial}{\partial r}\left[\rho_{\rm u}(v+v_{\rm u})^2+p\right] - S_{\rm u}\rho_{\rm u}\frac{Gm}{r^2}+\dot{Q}, \tag{5}$$

and

$$\frac{\partial}{\partial t}[S_{\rm d}\rho_{\rm d}(v-v_{\rm d})] = -S_{\rm d}\frac{\partial}{\partial r}\left[\rho_{\rm d}(v-v_{\rm d})^2+p\right] - S_{\rm d}\rho_{\rm d}\frac{Gm}{r^2}-\dot{Q}. \tag{6}$$

Here, G is the Newton constant of gravitation, m is the total mass contained inside the sphere of radius r

$$m = \int_0^r (S_{\rm u}\rho_{\rm u}+S_{\rm d}\rho_{\rm d})\,{\rm d}r', \tag{7}$$

and $\dot{Q}$ is the momentum per unit mass and unit time exchanged in this shell from the upward to the downward moving fluid.

2.1.3 Energy

The rate of change of energy is the advective plus diffusive energy flux, in the radial direction and sideways, added to the work of pressure forces and heat generation:

$$\frac{\partial}{\partial t}(S_{\rm u}\rho_{\rm u}e_{\rm u}) = -\frac{\partial}{\partial r}[S_{\rm u}\rho_{\rm u}e_{\rm u}(v+v_{\rm u})] - p\frac{\partial}{\partial r}[S_{\rm u}(v+v_{\rm u})] - p\frac{\partial}{\partial t}(S_{\rm u}) + S_{\rm u}\rho_{\rm u}\left(\epsilon_{\rm u}-\frac{\partial L_{\rm u}}{S_{\rm u}\rho_{\rm u}\partial r}\right)+\dot{E} \tag{8}$$

and

$$\frac{\partial}{\partial t}(S_{\rm d}\rho_{\rm d}e_{\rm d}) = -\frac{\partial}{\partial r}[S_{\rm d}\rho_{\rm d}e_{\rm d}(v-v_{\rm d})] - p\frac{\partial}{\partial r}[S_{\rm d}(v-v_{\rm d})] - p\frac{\partial}{\partial t}(S_{\rm d}) + S_{\rm d}\rho_{\rm d}\left(\epsilon_{\rm d}-\frac{\partial L_{\rm d}}{S_{\rm d}\rho_{\rm d}\partial r}\right)-\dot{E}. \tag{9}$$

Here, L is the luminosity carried by the conductive and radiative processes (in the radial direction):

$$L_{\rm u} = -\frac{S_{\rm u}c}{3\kappa_{\rm u}}\frac{\partial E_{\rm u}}{\partial r} \tag{10}$$

$$= \rho_{\rm u}c_{\rm Pu}\chi_{\rm u}\frac{\partial T_{\rm u}}{\partial r}S_{\rm u}, \tag{11}$$

and

$$L_{\rm d} = -\frac{S_{\rm d}c}{3\kappa_{\rm d}}\frac{\partial E_{\rm d}}{\partial r} \tag{12}$$

$$= \rho_{\rm d}c_{\rm Pd}\chi_{\rm d}\frac{\partial T_{\rm d}}{\partial r}S_{\rm d}. \tag{13}$$

Here, E is the radiative energy per unit of volume given by the equations of state, c is the speed of light, κ is the flux weighted total opacity for the conductive and radiative processes (in the diffusion approximation), χ is the thermal diffusion coefficient and $c_{\rm P}$ is the heat capacity at constant pressure.

The net energy production per unit time and per unit mass is ϵ. This can include the nuclear energy production if the rest mass energy is not yet included in the specific energy, and it includes the neutrino losses. The energy per unit time and per unit radius exchanged in the shell from the upward to the downward moving fluid is $\dot{E}$.

2.1.4 Chemistry

The rate of change of a species is its flux in the radial direction and sideways added to its chemical rate of change:

$$\frac{\partial}{\partial t}(S_{\rm u}\rho_{\rm u}\boldsymbol{N}_{\rm u}) = -\frac{\partial}{\partial r}[S_{\rm u}\rho_{\rm u}\boldsymbol{N}_{\rm u}(v+v_{\rm u})] + S_{\rm u}\rho_{\rm u}\boldsymbol{R}_{\rm u}+\dot{\boldsymbol{N}} \tag{14}$$

and

$$\frac{\partial}{\partial t}(S_{\rm d}\rho_{\rm d}\boldsymbol{N}_{\rm d}) = -\frac{\partial}{\partial r}[S_{\rm d}\rho_{\rm d}\boldsymbol{N}_{\rm d}(v-v_{\rm d})] + S_{\rm d}\rho_{\rm d}\boldsymbol{R}_{\rm d}-\dot{\boldsymbol{N}}. \tag{15}$$

Here, $\dot{N}_j$ represents the mass of species j per unit radius and per unit time exchanged in the shell from the downward moving fluid to the upward moving fluid, and R_j is the rate of change of species j per unit time and unit mass due to nuclear reactions.

2.2 Equations for the mean fluid

When we sum up each pair of the above equations we obtain the equations for the variation of the mean mass, energy and momentum. We first derive these equations in their volumic form and then gather them all in a more familiar specific form.

2.2.1 Mass

The mean density ρ is defined by

$$\rho S = \rho_{\rm u}S_{\rm u}+\rho_{\rm d}S_{\rm d}. \tag{16}$$

The equation for mass conservation then reads

$$\frac{\partial}{\partial t}(S\rho) = -\frac{\partial}{\partial r}(S\rho v), \tag{17}$$

which is the unchanged continuity equation for the mean fluid.

2.2.2 Momentum

The mean equation for momentum is more complex and involves several additional terms

$$\frac{\partial}{\partial t}(S\rho v) = -S\frac{\partial}{\partial r}(\rho v^2+p) - S\rho\frac{Gm}{r^2}+S\rho X_{\rm conv}, \tag{18}$$

where

$$S\rho X_{\rm conv} = X_1+X_2+X_3, \tag{19}$$

$$X_1 = -S_{\rm u}\frac{\partial}{\partial r}\left(\rho_{\rm u}v_{\rm u}^2\right) - S_{\rm d}\frac{\partial}{\partial r}\left(\rho_{\rm d}v_{\rm d}^2\right), \tag{20}$$

$$X_2 = -2v\left[S_{\rm u}\frac{\partial}{\partial r}(\rho_{\rm u}v_{\rm u}) - S_{\rm d}\frac{\partial}{\partial r}(\rho_{\rm d}v_{\rm d})\right] \tag{21}$$

and

$$X_3 = (\rho_{\rm d}-\rho_{\rm u})v^2\left(\frac{S_{\rm u}}{S}\frac{\partial S_{\rm d}}{\partial r} - \frac{S_{\rm d}}{S}\frac{\partial S_{\rm u}}{\partial r}\right), \tag{22}$$

where we have used equation (1). Usually, the evolution of the star is quasi-static, which means that v is negligible. Then only X_1 should be retained because X_2 and X_3 are of order 1 and 2 in v.

X_1 accounts for a convective pressure support in the flow. It slightly changes the hydrostatic equilibrium but is usually negligible in the subsonic regime.

2.2.3 Energy

We define the mean specific energy as

$$S\rho e = \rho_{\rm u} S_{\rm u} e_{\rm u} + \rho_{\rm d} S_{\rm d} e_{\rm d} \tag{23}$$

and the mean energy generation per unit mass ϵ in the same way as

$$\rho\epsilon = \rho_{\rm u} S_{\rm u} \epsilon_{\rm u} + \rho_{\rm d} S_{\rm d} \epsilon_{\rm d}. \tag{24}$$

The radiative and conductive luminosity is given by

$$L = L_{\rm u} + L_{\rm d}. \tag{25}$$

Then the equation for the mean energy is rather simple and includes only two additional terms

$$\frac{\partial}{\partial t}(S\rho e) = -\frac{\partial}{\partial r}(S\rho e v) - p\frac{\partial}{\partial r}(Sv) + S\rho\left(\epsilon - \frac{\partial L + L_{\rm conv}}{S\rho\partial r}\right) + S\rho W_{\rm conv}, \tag{26}$$

where

$$L_{\rm conv} = \dot{m}(h_{\rm u} - h_{\rm d}) = \frac{1}{2}S\rho u(h_{\rm u} - h_{\rm d}), \tag{27}$$

$$W_{\rm conv} = \frac{1}{2}u\left(\frac{1}{\rho_{\rm u}} - \frac{1}{\rho_{\rm d}}\right)\frac{\partial p}{\partial r}. \tag{28}$$

Here $h = e + p/\rho$ is the enthalpy and $u = 2\dot{m}/(S\rho)$ is the mean convective velocity.

Note that the convective luminosity $L_{\rm conv}$ naturally appears as an enthalpy flux.

The term $W_{\rm conv}$ can be interpreted as the work done by the buoyancy forces. Because the pressure is decreasing upward, and usually $\rho_{\rm u} < \rho_{\rm d}$, it always provides a sink term for the energy.

Note that this approach cannot be used to write a general equation for the entropy because the thermodynamic relations are only valid for the up and down streams separately but not for the mean fluid. We will derive an entropy equation later in the one-flow limit (see Section 4.3).

2.2.4 Chemistry

The equation for the mean abundances is

$$\frac{\partial}{\partial t}(S\rho \boldsymbol{N}) = -\frac{\partial}{\partial r}(S\rho \boldsymbol{N} v) + S\rho \boldsymbol{R} - S\rho\frac{\partial}{\partial m}\boldsymbol{F}_{\rm conv}, \tag{29}$$

where

$$S\rho \boldsymbol{N} = S_{\rm u}\rho_{\rm u}\boldsymbol{N}_{\rm u} + S_{\rm d}\rho_{\rm d}\boldsymbol{N}_{\rm d} \tag{30}$$

and

$$\boldsymbol{F}_{\rm conv} = \dot{m}(\boldsymbol{N}_{\rm u} - \boldsymbol{N}_{\rm d}), \tag{31}$$

which we interpret as a diffusion flow in Section 2.5.

2.2.5 Mean specific equations

We now summarize all of the mean equations derived in the previous section in their specific form (i.e. per unit mass), where we use the notation

$$\frac{{\rm D}x}{{\rm D}t} = \frac{\partial x}{\partial t} + v\frac{\partial x}{\partial r}$$

and

$$\mathrm{div}(x) = \frac{1}{S}\frac{\partial Sx}{\partial r}:$$

$$\frac{1}{\rho}\frac{{\rm D}\rho}{{\rm D}t} = -\mathrm{div}(v), \tag{32}$$

$$\frac{{\rm D}v}{{\rm D}t} - 2\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} - \frac{Gm}{r^2} + X_{\rm conv}, \tag{33}$$

$$\frac{{\rm D}e}{{\rm D}t} + p\frac{{\rm D}}{{\rm D}t}\left(\frac{1}{\rho}\right) = \epsilon - \frac{\partial L}{\partial m} - \frac{\partial L_{\rm conv}}{\partial m} + W_{\rm conv} \tag{34}$$

and

$$\frac{{\rm D}\boldsymbol{N}}{{\rm D}t} = \boldsymbol{R} - \frac{\partial}{\partial m}\boldsymbol{F}_{\rm conv} \tag{35}$$

with

$$X_{\rm conv} \simeq -\frac{1}{S\rho}\left[S_{\rm u}\frac{\partial}{\partial r}\left(\rho_{\rm u}v_{\rm u}^2\right) + S_{\rm d}\frac{\partial}{\partial r}\left(\rho_{\rm d}v_{\rm d}^2\right)\right], \tag{36}$$

$$L_{\rm conv} = \dot{m}(h_{\rm u} - h_{\rm d}), \tag{37}$$

$$W_{\rm conv} = \frac{1}{2}u\left(\frac{1}{\rho_{\rm u}} - \frac{1}{\rho_{\rm d}}\right)\frac{\partial p}{\partial r}, \tag{38}$$

and

$$\boldsymbol{F}_{\rm conv} = \dot{m}(\boldsymbol{N}_{\rm u} - \boldsymbol{N}_{\rm d}). \tag{39}$$

These equations are the usual equations of radiative stellar evolution with additional terms due to the differential motions in the two fluids. The latter terms depend on the convective velocities $v_{\rm u}$ and $v_{\rm d}$, as well as on differences between quantities in the two flows. We derive equations for the mean convective velocities and those differences in the one-flow limit in Section 2.4.

2.3 Specific equations

We first rewrite the conservation equations in their specific form. This will allow the derivation of the equations in the one-flow limit when we take the difference between the specific equations for the up and down motions.

2.3.1 Momentum

$$\frac{\partial}{\partial t}(v + v_{\rm u}) + (v + v_{\rm u})\frac{\partial}{\partial r}(v + v_{\rm u}) - (v + v_{\rm u})^2\frac{1}{S_{\rm u}}\frac{\partial S_{\rm u}}{\partial r} = -\frac{1}{\rho_{\rm u}}\frac{\partial p}{\partial r} - \frac{Gm}{r^2} + \frac{1}{S_{\rm u}\rho_{\rm u}}[\dot{Q} - \dot{M}(v + v_{\rm u})] \tag{40}$$

and

$$\frac{\partial}{\partial t}(v - v_{\rm d}) + (v - v_{\rm d})\frac{\partial}{\partial r}(v - v_{\rm d}) - (v - v_{\rm d})^2\frac{1}{S_{\rm d}}\frac{\partial S_{\rm d}}{\partial r} = -\frac{1}{\rho_{\rm d}}\frac{\partial p}{\partial r} - \frac{Gm}{r^2} - \frac{1}{S_{\rm d}\rho_{\rm d}}[\dot{Q} - \dot{M}(v - v_{\rm d})]. \tag{41}$$

2.3.2 Energy

$$\frac{\partial e_{\rm u}}{\partial t} + (v + v_{\rm u})\frac{\partial e_{\rm u}}{\partial r} + p\left[\frac{\partial}{\partial t}\frac{1}{\rho_{\rm u}} + (v + v_{\rm u})\frac{\partial}{\partial r}\frac{1}{\rho_{\rm u}}\right] = \epsilon_{\rm u} - \frac{1}{S_{\rm u}\rho_{\rm u}}\frac{\partial L_{\rm u}}{\partial r} + \frac{1}{S_{\rm u}\rho_{\rm u}}(\dot{E} - \dot{M}h_{\rm u}) \tag{42}$$

and

$$\frac{\partial e_{\rm d}}{\partial t} + (v - v_{\rm d})\frac{\partial e_{\rm d}}{\partial r} + p\left[\frac{\partial}{\partial t}\frac{1}{\rho_{\rm d}} + (v - v_{\rm d})\frac{\partial}{\partial r}\frac{1}{\rho_{\rm d}}\right] = \epsilon_{\rm d} - \frac{1}{S_{\rm d}\rho_{\rm d}}\frac{\partial L_{\rm d}}{\partial r} - \frac{1}{S_{\rm d}\rho_{\rm d}}(\dot{E} - \dot{M}h_{\rm d}). \tag{43}$$

2.3.3 Chemistry

$$\frac{\partial \boldsymbol{N}_{\rm u}}{\partial t} + (v + v_{\rm u})\frac{\partial \boldsymbol{N}_{\rm u}}{\partial r} = \boldsymbol{R}_{\rm u} + \frac{1}{S_{\rm u}\rho_{\rm u}}(\dot{\boldsymbol{N}} - \dot{M}\boldsymbol{N}_{\rm u}) \tag{44}$$

and

$$\frac{\partial \boldsymbol{N}_{\rm d}}{\partial t} + (v - v_{\rm d})\frac{\partial \boldsymbol{N}_{\rm d}}{\partial r} = \boldsymbol{R}_{\rm d} - \frac{1}{S_{\rm d}\rho_{\rm d}}(\dot{\boldsymbol{N}} - \dot{M}\boldsymbol{N}_{\rm d}). \tag{45}$$

2.4 Difference equations in the one-stream limit

In the following we make the approximation that the relative difference of a quantity between the two streams is small. For each quantity x we write $x_{\rm u} = x + \Delta x$ and $x_{\rm d} = x - \Delta x$ with $\Delta x \ll x$. We note that the previously defined mean convective velocity u is in fact the arithmetic mean of $v_{\rm u}$ and $v_{\rm d}$. Hence, we write $v_{\rm u} = u + \Delta u$ and $v_{\rm d} = u - \Delta u$. In contrast, we keep the former definition for S as the total surface of the shell of radius r so that $S_{\rm u} = (1/2)S + \Delta S$ and $S_{\rm d} = (1/2)S - \Delta S$. We then compute the difference of the specific equations (40)–(45) between the two fluids and neglect second-order terms in the Δ quantities.

In this approximation, the averages defined in Section 2.2 are all arithmetic means. We also note that the standard thermodynamic relations hold for the Δ quantities, which greatly helps when deriving the entropy equation (see Section 4.3).

Equation (2) yields

$$2\frac{\Delta S}{S} + \frac{\Delta\rho}{\rho} + \frac{\Delta u}{u} = 0 \tag{46}$$

and this allows us to eliminate ΔS in what follows. It can be thought of as the equation governing the convective motions with u as the convective velocity.

2.4.1 Mass

Taking the difference between equations (3) and (4) and using equation (32) we find

$$\frac{\rm D}{{\rm D}t}\left(\frac{\Delta u}{u}\right) = \frac{1}{S\rho}\frac{\partial}{\partial r}(S\rho u) - \frac{2\dot{M}}{S\rho}. \tag{47}$$

This equation describes the evolution of the asymmetry Δu of the drift motions relative to the mean velocity v.

2.4.2 Momentum

The difference between equations (40) and (41) yields

$$\frac{{\rm D}u}{{\rm D}t} + u\left[\frac{\partial(v + 2\Delta u)}{\partial r} - 4\frac{v + \Delta u}{r}\right] = \frac{\Delta\rho}{\rho^2}\frac{\partial p}{\partial r} - (v^2 + u^2)\frac{\partial}{\partial r}\left(\frac{\Delta\rho}{\rho}\right) - v^2\frac{\partial}{\partial r}\left(\frac{\Delta u}{u}\right) + \frac{2}{S\rho}[\dot{Q} - \dot{M}(v + 2\Delta u)]. \tag{48}$$

This equation describes the time evolution of the convective velocity u. The main source term is the acceleration due to the buoyancy force while the main sink term comes from the momentum exchange term between the two flows.

Apart from the Δu terms

$$-(v^2 + u^2)\frac{\partial}{\partial r}\left(\frac{\Delta\rho}{\rho}\right)$$

is the only non-local term, i.e. it is the only term that involves spatial derivatives of convective properties.

2.4.3 Energy

The difference between equations (42) and (43) yields

$$\frac{{\rm D}\Delta e}{{\rm D}t} - p\frac{\rm D}{{\rm D}t}\left(\frac{\Delta\rho}{\rho^2}\right) + u\left(\frac{\partial e}{\partial r} + p\frac{\partial}{\partial r}\frac{1}{\rho}\right) = \Delta\epsilon + 2\frac{\Delta u}{u}\frac{\partial L}{\partial m} + 2\frac{\partial \Delta L}{\partial m} + \frac{2}{S\rho}(\dot{E} - \dot{M}h). \tag{49}$$

This equation implicitly determines the time evolution of the temperature difference between the two flows, as we show in Section 3.1. Note that the $\Delta\epsilon$ term may help to either increase or decrease such a temperature difference, depending on the temperature dependence of the energy generation rate. The other non-local term in the energy equation, $2(\partial\Delta L/\partial m)$, accounts for differential thermal diffusion along the two columns of fluid.

2.4.4 Chemistry

The difference between equations (14) and (15) yields

$$\frac{{\rm D}\Delta\boldsymbol{N}}{{\rm D}t} + u\frac{\partial \boldsymbol{N}}{\partial r} = \Delta\boldsymbol{R} + \frac{2}{S\rho}(\dot{\boldsymbol{N}} - \dot{M}\boldsymbol{N}). \tag{50}$$

This equation describes the evolution of the chemical composition difference between the two flows. In some situations, the temperature dependence of the nuclear reaction rates may affect this difference through the $\Delta\boldsymbol{R}$ term.

2.5 Model for the exchange terms

All horizontal motions and transport phenomena are modelled through the exchange terms. These need to be specified to close the two systems of equations (32)–(35) and (47)–(50). With these, the equations provide a complete time-dependent model for convection which guarantees the conservation of mass, momentum, energy and chemical transformations. Most of the existing convective models provide approximations for these exchange terms. Here we present a simple, somewhat ad hoc, but physically plausible choice for these exchange terms:

$$\dot{Q} = -(v_{\rm u} + v_{\rm d})\dot{m}/\lambda + \dot{M}v, \tag{51}$$

$$\dot{E} = (h_{\rm d} - h_{\rm u})\dot{m}/\lambda + \beta\frac{\chi}{u\lambda}c_p(T_{\rm d} - T_{\rm u})\frac{\dot{m}}{\lambda} + \dot{M}h, \tag{52}$$

and

$$\dot{\boldsymbol{N}} = (\boldsymbol{N}_{\rm d} - \boldsymbol{N}_{\rm u})\dot{m}/\lambda + \dot{M}\boldsymbol{N}. \tag{53}$$

The first terms in these expressions account for momentum, energy and chemical exchanges without any net transfer of mass. They are designed to mix the two fluids on a length-scale λ. The second term in the energy exchange equation (52) accounts for horizontal

heat diffusion across the edges of the streams. The parameter β is a form factor which fixes the ratio of the perimeter of the streams to their separation times S/λ^2. We use $\beta = 9/2$ to recover the exact formulation of classical mixing length theory (MLT; Böhm-Vitense 1958, as presented in Kippenhahn & Weigert 1990). The last terms in expressions (51)–(53) are the fluxes due to a net transfer of mass $\dot{M}$ from one stream to the other. Such transfer must exist at the outer boundaries of a convective zone where the fluid effectively makes a U-turn.

There is no obvious, simple physical prescription for $\dot{M}$ so we make the assumption that $u = v_{\rm u} = v_{\rm d}$ and convective motions are symmetric with respect to the mean velocity v. This approximation is in fact implicit in almost all convective models to date. The term $\dot{M}$ is then given by equations (2)–(4) as

$$\dot{M} = \frac{\partial \dot{m}}{\partial r}. \tag{54}$$

We now have a complete time-dependent description of the convective properties of the flow and can investigate the characteristics of this model.

3 COMPARISON WITH EXISTING MODELS FOR CONVECTION

In this section we consider the approximations made in a number of theories of convection. Using the same assumptions, we derive the equations for the convective motions in our framework and emphasize the characteristics that are peculiar to our formalism.

3.1 Mixing length theory

MLT assumes a stationary state for convection and makes the quasi-static approximation, $v \simeq 0$. In this case, all D/Dt terms can be set equal to zero in the difference equations. In addition, we set Δu, $\Delta\epsilon$, ΔL and $\Delta \boldsymbol{R}$ to zero because the processes that lead to these terms are usually neglected in MLTs. We also neglect the non-local term in the momentum difference equation. The difference equations then simplify to

$$\frac{\Delta\rho}{\rho^2}\frac{\partial p}{\partial r} = 2\frac{u^2}{\lambda}, \tag{55}$$

$$\frac{\partial e}{\partial r} + p\frac{\partial}{\partial r}\frac{1}{\rho} = -2\frac{\Delta h}{\lambda} - 2\beta\frac{\chi c_p \Delta T}{u\lambda^2} \tag{56}$$

and

$$\frac{\partial \boldsymbol{N}}{\partial r} = -2\frac{\Delta \boldsymbol{N}}{\lambda}. \tag{57}$$

Note that equation (55) is not exactly the same as in classical MLT: there is usually a factor of 1/2 multiplying the left-hand side of this equation to account for the fact that half of the work done by the buoyancy force is used to push aside the surrounding medium when a convective element rises.

From these equations we extract the convective velocity, the temperature and chemical differences between the two streams (see Appendix A1.1 for a detailed derivation). These can now be used to express the convective terms in the mean equations

$$X_{\rm conv} = -\frac{1}{\rho}\frac{\partial}{\partial r}(\rho u^2), \tag{58}$$

$$\dot{m} = \frac{1}{2}S\rho u, \tag{59}$$

$$L_{\rm conv} = \dot{m}(c_p\Delta T + \boldsymbol{\mu}' \cdot \Delta \boldsymbol{N}), \tag{60}$$

$$W_{\rm conv} = -\frac{u^3}{\lambda} \tag{61}$$

and

$$\boldsymbol{F}_{\rm conv} = 2\dot{m}\Delta \boldsymbol{N} \tag{62}$$

where $\boldsymbol{\mu}' = T(\partial s/\partial \boldsymbol{N})_{T,p} + \boldsymbol{\mu}$ and $\boldsymbol{\mu}$ is the chemical potential (see Appendix A1.1).

With these equations, we have derived a MLT that is consistent with chemistry. While the excess temperature equation is unchanged, the convective velocity now depends on the chemical stratification (through the cubic A5) and so does the convective luminosity. There is also an additional work term due to the fact that we assume a reversible process for the momentum exchange (see Section 3.3). We also obtain an explicit change in the condition of hydrostatic equilibrium owing to convection. Finally, convection naturally appears as a diffusion process for the chemistry.

3.2 Unno (1967)

If we include the D/Dt terms in equations (55) and (56) and further neglect the chemistry dependence of the convective luminosity and velocity, we immediately recover the same time-dependent version of MLT as Unno (1967) for the excess temperature and the convective velocity.

3.3 Kuhfuss (1986)

Kuhfuss (1986) only computes the evolution equation for the convective velocity. He uses a diffusion model to compute the correlations between velocity perturbations and any other perturbation. This model is recovered in our formalism if we set D/D$t \equiv 0$ and $\Delta \boldsymbol{R} = \Delta\epsilon = \Delta L = 0$ in the difference equations for energy and chemistry.

Furthermore, each term of his equation (25c) for the convective kinetic energy corresponds to one term in our equation (48), except that his non-local term $(1/\langle\rho\rangle)\,{\rm div}\,\boldsymbol{j}_t$ is different from ours (see Section 3.5), and we do not account for his viscous terms (we assumed an inviscid fluid).

Finally, if we compare our equation for the average internal energy of the gas, we note that our model misses the heat production owing to dissipation of convective motions. This is due to the fact that we assume a reversible exchange of momentum so that there is no associated heat production. In contrast, Kuhfuss (1986) assumes viscous dissipation for the convective motions.

3.4 Eggleton (1983)

Eggleton (1983) uses his rule of thumb to average the hydrodynamical equations in order to obtain evolution equations for the mean fluid and the perturbed quantities. He then obtains a full set of equations that can be identified with our mean and difference equations. Our formalism agrees fairly well with this rule of thumb and hence provides a more physical basis for it.

We can recover almost the same equations if we set $\Delta u = 0$. The local models differ only in the mean energy equation because Eggleton neglects the thermal part of the chemical potentials. However, those terms are important when beta decays or electron captures occur in degenerate matter.

As for the non-local terms, $\partial \Delta L/\partial m$ in the energy difference equation takes the same form of the term in equation (41) of Eggleton (1983) which accounts for the thermal diffusion through the front and back of the eddies. However, our non-local term in the momentum difference equation does not agree with equation (42) of Eggleton (1983) for the velocity perturbation.

An interesting point is that Eggleton (1983) found a term similar to our $\Delta \boldsymbol{R}$ term. We have shown that an additional term $\Delta\epsilon$ enters into the energy equation. These terms account for differential reactivity in the two streams. However, our stationary computations show that these effects are negligible as far as the convective Urca process is concerned (see Section 6).

3.5 Grossman et al. (1993)

Grossman et al. (1993) use the Boltzmann equation coupled with dynamical equations to compute a hierarchy of moments for the hydrodynamical equations. In their framework, our formalism can be recovered if we specify the distribution function of the blobs as

$$f_A(t, z, v, T) = \frac{1}{S}[S_\mathrm{u}\rho_\mathrm{u}\delta(v + v_\mathrm{u}, T_\mathrm{u}) + S_\mathrm{d}\rho_\mathrm{d}\delta(v - v_\mathrm{d}, T_\mathrm{d})]. \quad (63)$$

This allows a direct comparison between the two formalisms. With this definition, our variables v, ρ and T correspond to their variables $\bar{v}$, $\bar{\rho}$ and $\bar{T}$.

The zeroth-order equations of their hierarchy can be directly compared with our mean equations. The velocity equation is found to differ by terms negligible in the subsonic regime. Their temperature equation without chemistry has an additional source term due to viscous dissipation just as Kuhfuss (1986). Their temperature equation with chemistry misses correction terms in the luminosity, the work term and the source term. This is due to their use of a dynamical equation for the entropy which does not account for the changes owing to chemical evolution. This suggests that energy conservation may not hold in their case when chemistry is included.

Their higher-order equations can also be compared with our difference equations, although this is less straightforward. We made such a comparison but only for the velocity difference equation. We then obtain the same non-local term as Kuhfuss (1986): their $(1/\bar{\rho})(\partial/\partial z)(\bar{\rho}\bar{w}^3)$ corresponds to $(1/\langle\rho\rangle)\,\mathrm{div}\,\boldsymbol{j}_t$ of Kuhfuss (1986). If we substitute our distribution function f_A in this term, we obtain

$$\frac{2}{\rho}\frac{\partial}{\partial z}(\rho u^2 \Delta u), \quad (64)$$

which should be compared to our

$$2u^2\frac{\partial \Delta u}{\partial r} + u^3\frac{\partial}{\partial r}\frac{\Delta\rho}{\rho} \quad (65)$$

when $v = 0$. Our term contains $\Delta\rho$ and theirs contains the spatial derivative of the convective velocity. On the other hand, our treatments agree as far as the chemical dependence of the convective velocity is concerned.

4 CONVECTIVE URCA PROCESS

In this section we first present the basic nuclear reactions responsible for the Urca process. Because previous studies have mainly concentrated on the entropy equation, we derive it in our formalism and then examine the influence of the Urca nuclei on convection.

4.1 Urca reactions

Urca reactions involve pairs of nuclei of the form $({}^{Z+1}_{A}\mathrm{M}, {}^{Z}_{A}\mathrm{D})$ where A is an odd number. The member of a pair with an additional proton is called the mother (M), while the other one is referred to as the daughter (D). Electron capture and beta decay turn one into the other:

$$\text{electron capture: } \mathrm{M} + \mathrm{e}^- \rightarrow \mathrm{D} + \nu \quad (66)$$

$$\text{beta decay: } \mathrm{D} \rightarrow \mathrm{M} + \mathrm{e}^- + \bar{\nu}. \quad (67)$$

Tsuruta & Cameron (1970) give the reaction rates for electron capture and beta decay λ^+ and λ^- per nucleus. They also provide the corresponding neutrino losses L^+ and L^- per nucleus. The typical time-scale for Urca reactions is 10^5 s. These reaction rates depend mainly on the chemical potential μ_e of the electrons. This is mainly on the mass density ρ of the degenerate matter. Each Urca pair has a threshold energy μ_th above which significant electron captures can occur. We approximate the Coulomb corrections to these threshold energies in the same way as Gutiérrez et al. (1996). When $\mu_\mathrm{e} > \mu_\mathrm{th}$ (i.e. $\rho > \rho_\mathrm{th}$) electron captures quickly turn the Urca matter into daughter nuclei. When $\mu_\mathrm{e} < \mu_\mathrm{th}$ (i.e. $\rho < \rho_\mathrm{th}$) beta decays quickly turn it into mother nuclei. When $\mu_\mathrm{e} \simeq \mu_\mathrm{th}(\rho \simeq \rho_\mathrm{th})$ both reactions are significant and the Urca matter quickly evolves into a mixture of mother and daughter nuclei. A shell on which $\rho = \rho_\mathrm{th}$ is called an Urca shell.

When the Urca matter is in chemical equilibrium both reactions take place at the same rate, which is highest near the location of the Urca shell. Because both reactions emit neutrinos, this leads to neutrino cooling which is strongest at the Urca shell. When the Urca matter is far from equilibrium, nuclear heating takes place at the same time and usually dominates over the neutrino cooling (see Bruenn 1973, and Section 4.3.1).

The pair ${}^{23}\mathrm{Na}/{}^{23}\mathrm{Ne}$ is a typical example and perhaps the most active Urca pair in massive white dwarfs. Its threshold energy is $\mu_\mathrm{th} = 4.38$ MeV, which corresponds to $\rho_\mathrm{th} = 1.7 \times 10^9$ g cm^{-3} in purely degenerate matter. This density is slightly below the density for carbon ignition in a C+O white dwarf accreting at a rate of 10^{-7} $\mathrm{M}_\odot$ yr^{-1}. As a consequence, the growing convective core soon engulfs the corresponding Urca shell after carbon has ignited in the centre.

Here we assume that the stellar matter contains a fixed number N_U of Urca nuclei (per unit mass) of a given pair. We define N_M and N_D as the corresponding number of mother and daughter nuclei so that $N_\mathrm{U} = N_\mathrm{M} + N_\mathrm{D}$. Then the rate of change of mother and daughter nuclei per unit mass becomes

$$R_\mathrm{M} = -R_\mathrm{D} = -\lambda^+ N_\mathrm{M} + \lambda^- N_\mathrm{D}. \quad (68)$$

If we write $N^*_\mathrm{M} = N_\mathrm{U}\lambda^-/(\lambda^+ + \lambda^-) = N_\mathrm{U} - N^*_\mathrm{D}$ as the number of mother nuclei per unit mass at chemical equilibrium, we can rewrite the reaction rate as

$$R_\mathrm{M} = -(\lambda^+ + \lambda^-)\left(N_\mathrm{M} - N^*_\mathrm{M}\right). \quad (69)$$

Useful explicit expressions for the various thermodynamical properties of the Urca nuclei are

$$\mu_\mathrm{M} = (Z + 1)\mu_\mathrm{e} + kT(a + \ln N_\mathrm{M}), \quad (70)$$

$$\mu_\mathrm{D} = Z\mu_\mathrm{e} + kT(a + \ln N_\mathrm{D}), \quad (71)$$

$$\mu'_\mathrm{M} \simeq (Z + 1)\mu_\mathrm{e} + \frac{1}{2}kT, \quad (72)$$

$$\mu'_\mathrm{D} \simeq Z\mu_\mathrm{e} + \frac{1}{2}kT, \quad (73)$$

$$\mu''_\mathrm{M} = -(Z + 1)N_\mathrm{M}/N_\mathrm{e} \quad (74)$$

and

$$\mu''_{\rm D} = -ZN_{\rm D}/N_{\rm e}, \tag{75}$$

where $N_{\rm e}$ is the total number of electrons per unit mass, k is the Boltzmann constant and a is a combination of temperature and density logarithms (see Pols et al. 1995). The approximations for $\mu'_{\rm M}$ and $\mu'_{\rm D}$ are obtained by neglect of the electron and radiation contributions to the entropy. In the strongly degenerate case, the kT terms are usually negligible.

4.2 Convective velocity

In Appendix A (equation A8) we derive the approximation which relates the convective velocity to the temperature and chemical gradients

$$u = u_1\sqrt{\delta(\nabla - \nabla_a) - \boldsymbol{\mu}''\cdot\nabla_N}, \tag{76}$$

where the term depending on the Urca process is

$$\boldsymbol{\mu}''\cdot\nabla_N = -\frac{1}{N_{\rm e}}[(Z+1)N_{\rm M}\nabla_{\rm M} + ZN_{\rm D}\nabla_{\rm D}]. \tag{77}$$

Since $N_{\rm U}$ is assumed to be uniform, $N_{\rm M}\nabla_{\rm M} + N_{\rm D}\nabla_{\rm D} = 0$. Hence, relation (77) becomes

$$\boldsymbol{\mu}''\cdot\nabla_N = -\frac{N_{\rm M}}{N_{\rm e}}\nabla_{\rm M}. \tag{78}$$

Electron captures are much stronger in the centre of the star than in the outer regions, so the mother fraction generally increases outwards, $\boldsymbol{\mu}''\cdot\nabla_N > 0$ and the effect of the presence of Urca pairs is to reduce the convective velocity. In Sections 4.4 and 5 we show that this effect can actually be quite strong and may even inhibit convection. Note that, in the case where $N_{\rm U}$ is not uniform, the sign of the $\boldsymbol{\mu}''\cdot\nabla_N$ term may change.

4.3 Entropy equation

One of the biggest uncertainties in previous models for the convective Urca process was the form of the equation for the evolution of the entropy s in the presence of convection. Here we derive the entropy equation in the one-flow limit. The small Δ approximation allows us to use thermodynamic relations for the mean fluid and we can transform the left-hand side of equation (34) so that

$$T\frac{{\rm D}s}{{\rm D}t} + \boldsymbol{\mu}\frac{{\rm D}\boldsymbol{N}}{{\rm D}t} = \epsilon - \frac{\partial L}{\partial m} - \frac{\partial L_{\rm conv}}{\partial m} + W_{\rm conv}. \tag{79}$$

We then use equation (35) and the relation $\Delta h - \boldsymbol{\mu}\cdot\Delta\boldsymbol{N} = T\Delta s$ (because $\Delta p = 0$) to write the equation for the mean entropy as

$$T\frac{{\rm D}s}{{\rm D}t} = \epsilon' - \frac{\partial L}{\partial m} - \frac{\partial L'_{\rm conv}}{\partial m} + W'_{\rm conv}, \tag{80}$$

where we define

$$\epsilon' = \epsilon - \boldsymbol{\mu}\cdot\boldsymbol{R}, \tag{81}$$

$$L'_{\rm conv} = 2\dot{m}T\Delta s \tag{82}$$

and

$$W'_{\rm conv} = W_{\rm conv} - 2\dot{m}\Delta\boldsymbol{N}\cdot\frac{\partial\boldsymbol{\mu}}{\partial m}. \tag{83}$$

The entropy equation then takes a form that is similar to the energy equation but with different definitions for the net heating, the convective luminosity and the work. We next consider how these terms are affected by the chemical state of the Urca matter.

4.3.1 Net heat generation

The net heating due to Urca reactions is

$$\epsilon'_{\rm U} = \mu_{\rm th}R_{\rm M} - L^+N_{\rm M} - L^-N_{\rm D} - \mu_{\rm M}R_{\rm M} - \mu_{\rm D}R_{\rm D}. \tag{84}$$

In the very degenerate case, $\mu_{\rm M} \simeq (Z+1)\mu_{\rm e}$ and $\mu_{\rm D} \simeq Z\mu_{\rm e}$, we obtain

$$\epsilon'_{\rm U} = N_{\rm U}C + (N_{\rm M} - N^*_{\rm M})H, \tag{85}$$

where

$$C = -\frac{L^+\lambda^- + L^-\lambda^+}{\lambda^+ + \lambda^-} \tag{86}$$

$$H = -L^+ + L^- + (\mu_{\rm e} - \mu_{\rm th})(\lambda^+ + \lambda^-). \tag{87}$$

In Fig. 2 we plot $|C|$ and $|H|$ for the Urca pair ^{23}Na/^{23}Ne. The net Urca heating $\epsilon_{\rm U}{}'$ is the sum of two terms. At chemical equilibrium, only the first term remains. This is always negative and so causes cooling. The sign of the second term depends on the signs of $N_{\rm M} - N^*_{\rm M}$ and H. The term $N_{\rm M} - N^*_{\rm M}$ is likely to be positive below the Urca shell and negative above it. The value of H has the same sign except close to the Urca shell. Hence, the second term $(N_{\rm M} - N^*_{\rm M})H$ is positive, implying heating, except close to the Urca shell.

The relative magnitude of the second to the first term is proportional to the departure from chemical equilibrium $(N_{\rm M} - N^*_{\rm M})/N_{\rm U}$. It can be seen from Fig. 2 that the heating can be balanced by neutrino losses only if the system is close to chemical equilibrium because, generally, $|H| \gg |C|$. In a convective region mixing puts the Urca abundances slightly (or strongly if convection is very efficient) outside equilibrium. The amount of convective mixing is therefore crucial for computing the net Urca heating. Generally, the net effect is cooling close to the Urca shell and heating far away from it.

4.3.2 Convective luminosity

The contribution to $L'_{\rm conv}$ associated with the Urca pairs is

$$L'_{\rm U} = 2\dot{m}(\boldsymbol{\mu} - \boldsymbol{\mu}')\cdot\Delta\boldsymbol{N} = 2\dot{m}{\rm k}T\ln(N_{\rm D}/N_{\rm M})\Delta N_{\rm M}. \tag{88}$$

The quantity one needs to compare with $\epsilon'_{\rm U}$ and $W'_{\rm conv}$ is in fact the mass derivative of the luminosity $\partial L'_{\rm U}/\partial m$. This term is usually negligible in the very degenerate case. However, at convective

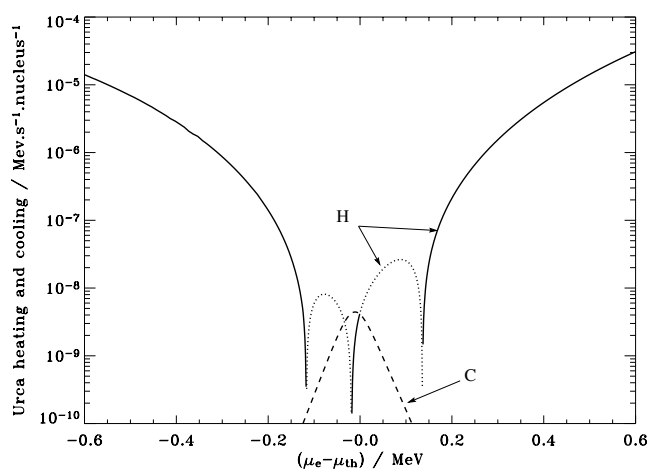

Figure 2. Functions $|H|$ (solid and dotted lines) and $-C$ (dashed line) for the ^{23}Na/^{23} Ne Urca pair at a temperature of 3×10^8 K. The solid portions of $|H|$ indicate where H and $N_{\rm M} - N^*_{\rm M}$ have the same sign (heating), and the dotted portions otherwise (cooling).

boundaries, the derivative $\partial\dot{m}/\partial m$ can be large and can make this term relatively more important.

4.3.3 Work term

Here we compute the contribution of the Urca pairs to the work term in the entropy equation

$$W'_{\rm U} = 2\dot{m}\Delta\, \boldsymbol{N}\cdot \frac{\partial \boldsymbol{\mu}}{\partial m} = 2\dot{m}\Delta N_{\rm M}\frac{\partial \mu_{\rm e}}{\partial m}. \tag{89}$$

Because the Urca matter is richer in mother nuclei in the outer parts of the star, $\Delta N_{\rm M} > 0$, and because the density is decreasing outwards, $\mu_{\rm e}$ is decreasing outwards. Hence, $W'_{\rm U}$ is always negative.

We note that $W'_{\rm U} = F_{\rm M}(\partial\mu_{\rm e}/\partial m)$ is identical to the work term Iben (1978b) designed for his computations, although he uses it in the energy equation and does not specify what convective velocity or luminosity he adopts.

Finally, no heat production is associated with our chemical exchange model. Indeed, we assume that a reversible process is responsible for the mixing of chemical species between both streams. An irreversible process for this chemical mixing could give rise to a heating term which could balance part (or all) of this additional work term.

4.4 Criterion for convection at the centre of the star

Let us now assume that we know the composition, temperature and density at the centre of a star. We can then derive a criterion for whether there is stationary convection and deduce an upper bound to the amount of mixing at the centre of a convective Urca core.

We consider a very small sphere of mass m at the centre of the star. In a stationary state, the total luminosity at the edge of this sphere must balance the energy production inside the sphere. If we assume that all the energy is carried out by convection we can write the convective luminosity

$$\epsilon m = L_{\rm conv} = 2\Delta h\dot{m} = 2(c_p T\Delta\ln T + \boldsymbol{\mu}'\cdot\Delta\boldsymbol{N})\dot{m}. \tag{90}$$

We obtain a similar expression for the net number of particles of each kind flowing away from this sphere

$$\boldsymbol{R}m = \boldsymbol{F}_{\rm conv} = 2\Delta\boldsymbol{N}\dot{m}. \tag{91}$$

Combining these two equations we obtain the density difference at the edge of the sphere

$$\Delta\ln\rho = \frac{m}{2\dot{m}}\left(-\delta\frac{\epsilon - \boldsymbol{\mu}'\cdot\boldsymbol{R}}{c_{\rm P}T} + \sum_j \frac{\mu''_j R_j}{N_j}\right). \tag{92}$$

A stationary convective state exists at the centre if and only if $\Delta\rho < 0$. This translates into an upper limit for the mother fraction at the centre of an Urca convective core. Using the relations $\mu'_j \simeq Z_j\mu_{\rm e}$ and $\mu''_j = -Z_jN_j/N_{\rm e}$, where $N_{\rm e}$ is the number of electrons per unit mass, we obtain

$$N_{\rm M} - N^*_{\rm M} < \delta\frac{\epsilon^*}{c_{\rm P}T}\left(\frac{\lambda^+ + \lambda^-}{N_{\rm e}} - \frac{\delta H}{c_{\rm P}T}\right)^{-1} = N_{\rm L} = AX_{\rm L}, \tag{93}$$

where ϵ^* is the net heating at Urca chemical equilibrium (equal to the heating from carbon burning at the centre) and $N_{\rm L}$ and $X_{\rm L}$ are defined by equation (93). $X_{\rm L}$ is the minimum mass fraction of Urca pairs which has a significant effect on convection. It generally has a fairly small value (see Fig. 3).

When $N_{\rm U} \gg N_{\rm L}$ the Urca pairs must be close to chemical equilibrium. In other words, the convective core can be only very slightly mixed. However, if they were in chemical equilibrium, carbon burning alone would produce a large buoyancy and hence drive strong mixing. Therefore, the Urca composition has to adjust itself to balance the heat from carbon burning and produce almost zero convective velocities. Hence, the inequality (93) is nearly an equality in practice and $N_{\rm M} - N^*_{\rm M} \simeq N_{\rm L}$, a result that is verified in our simulations of stationary convective cores (see Section 5.3).

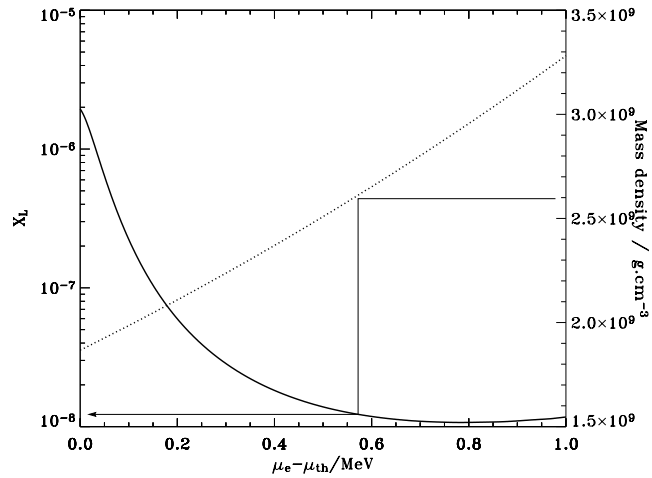

Figure 3. $X_{\rm L}$ (solid line) and ρ (dotted line) against $\mu_{\rm e} - \mu_{\rm th}$ for the ^{23}Na/^{23} Ne Urca pair at a temperature of 3×10^8 K.

5 STATIONARY CONVECTIVE URCA CORES

If we set all $\partial/\partial t$ derivatives equal to zero in equations (3)–(15), we obtain a system of coupled ordinary differential equations. If we specify the state variables at the centre of the star, we can integrate these equations outward, using a shooting method, and calculate the hydrostatic profile of the star.

To obtain a guess for the state variables at the centre, we run a very simple time-dependent model of an accreting white dwarf without Urca nuclei. We then compute the stationary state of stationary convective cores for different Urca compositions and different versions of our convective model.

5.1 Time-dependent model

We used the Eggleton (1971) stellar evolution code to calculate the evolution of a white dwarf, composed entirely of ^{12}C, ^{16}O and ^{20}Ne (with mass fractions ^{12}C = 0.25, ^{16}O = 0.73 and ^{20}Ne = 0.02) accreting matter at a rate of 10^{-7} M$_\odot$ yr^{-1}. The initial mass was taken as 1 M$_\odot$. Only the carbon-burning reaction ^{12}C(^{12}C,α)^{20}Ne immediately followed by ^{12}C(α,γ)^{16}O was taken into account. The convective model used in the Eggleton code is standard MLT and we use the approximate equation of state of Pols et al. (1995) assuming complete ionization.

We stop the computation during the carbon flash when the convective core has reached a mass of 0.4 M$_\odot$. We plot the convective velocity of this core against mass in Fig. 4. At this point, the threshold density for the ^{23}Na/^{23}Ne Urca pair is in the middle of the convective core. The central density is 2.6×10^9 g cm^{-3} and the central temperature is 3.1×10^8 K.

If we were to integrate the equations of hydrostatic equilibrium from this central state using MLT, we would obtain a fully convective star. The reason is that the core is actually being heated and that the term T (Ds/Dt) is non-zero. This term is negligible in the

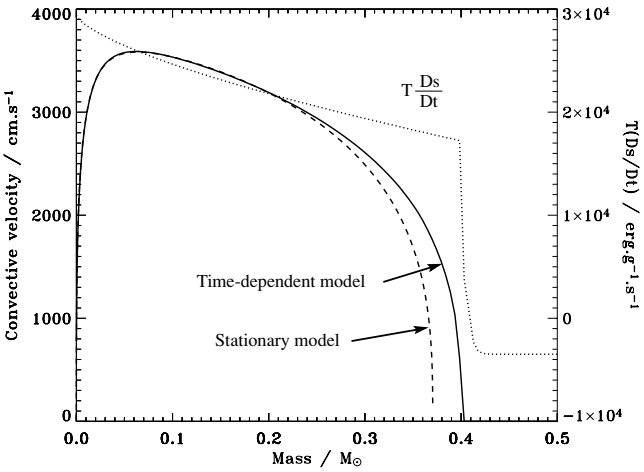

Figure 4. Convective velocity profile through the core. The solid line is for the time-dependent simulation and the dashed line is for the hydrostatic simulation. We also plot T (Ds/Dt) in the time-dependent simulation (dotted line).

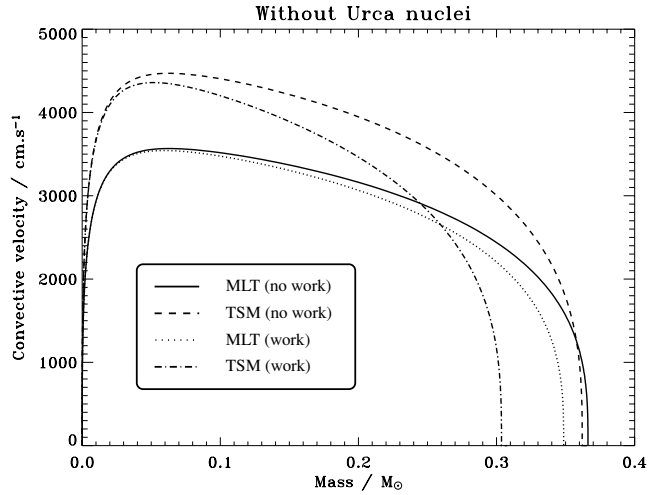

Figure 5. Convective velocity profile of different models of stationary cores without Urca nuclei. Solid and dashed lines are for MLT and TSM models with viscous dissipation of momentum. Dotted and dash-dotted lines are for MLT and TSM models with the work term $W_{\rm conv}$.

carbon-burning region but is significant in the outer part of the convective region. However, the effect of convection is to homogenize the entropy profile and so T (Ds/Dt) is rather uniform in the convective region. We can therefore use its value to offset the nuclear heating when we compute the hydrostatic profile. This brings the stationary convective profile very close to the time-dependent one (see Fig. 4).

5.2 Shooting method

Setting all the $\partial/\partial t$ terms to zero in the equations for the two streams, we obtain a system of coupled ordinary differential equations, which we integrate numerically from the centre of the star. As central boundary conditions for temperature, density, C, O and ^{20}Ne mass fractions we take the results of the time-dependent model. For each value of $N_{\rm U}$, we make an initial guess for the number density $N_{\rm M}$. The estimate for $N_{\rm M}$ is then iteratively improved by successive outward integrations until the condition $\Delta N_{\rm M} = 0$ is satisfied at the outer edge of the convective region.

5.3 Results

We compute stationary convective cores for different mass fractions of Urca pairs $X_{\rm U} = 23 \times N_{\rm U} = 0, 10^{-12}, 10^{-9}, 10^{-8}, 10^{-6}$ and 10^{-3}. A value of $X_{\rm U} = 10^{-3}$ would require a very efficient conversion of ^{20}Ne into Urca pairs and hence gives a reasonable upper limit for the possible abundance of Urca pairs.

5.3.1 Without Urca nuclei

For a pure C+O+^{20}Ne mixture, we compare the velocity profiles of the stationary convective core given by MLT and the two-stream model (TSM). In the TSM, we first set the work term $W_{\rm conv}$ equal to 0. Fig. 5 shows that the velocities differ by a factor of $\sqrt{2}$. This is the only difference between the two models and it can be traced back to the factor 2 in equation (55). Indeed, MLT has a factor 4 instead because it assumes that half of the work done by the buoyancy forces is used to push aside the surrounding medium when a convective element rises.

When we put the work term back in MLT or TSM, the resulting convective core shrinks. Less convection is needed to carry out the carbon-burning energy. This is not surprising because adding the work term is equivalent to suppressing the viscous heat produced by the dissipation of the drift motions.

5.3.2 Very low Urca abundance

For $X_{\rm U} < 10^{-9}$, the Urca nuclei do not have an impact on the convective velocity. They are mixed passively through the convective region. Fig. 6 shows the relative abundance of mother nuclei $N_{\rm M}/N_{\rm U}$ in both streams. Mother nuclei come from above the Urca shell. They capture electrons as they descend below the Urca shell and are converted into daughter nuclei. As these rise back above the Urca shell, they emit electrons and the number of daughter nuclei rises again. Finally, they cycle back down through the Urca shell.

The maximum relative difference of composition between both fluids is 17 per cent. This suggests that the one-stream approximation may well be adequate.

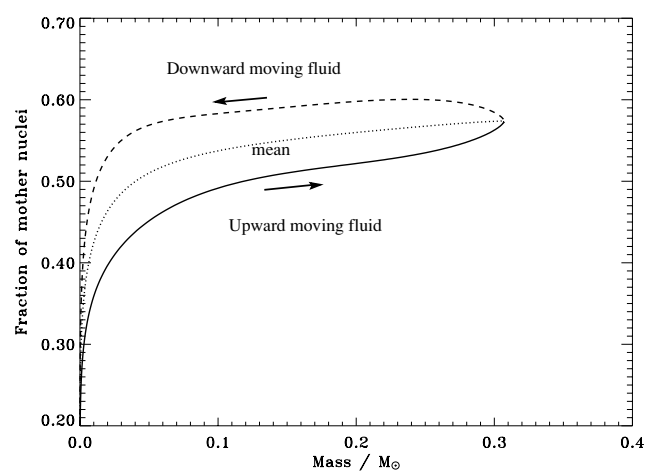

Figure 6. Fraction of mother nuclei $N_{\rm M}/N_{\rm U}$ against mass in the upward and downward moving streams for $X_{\rm U} < 10^{-9}$. The mean is also indicated. The model is the TSM including the work term.

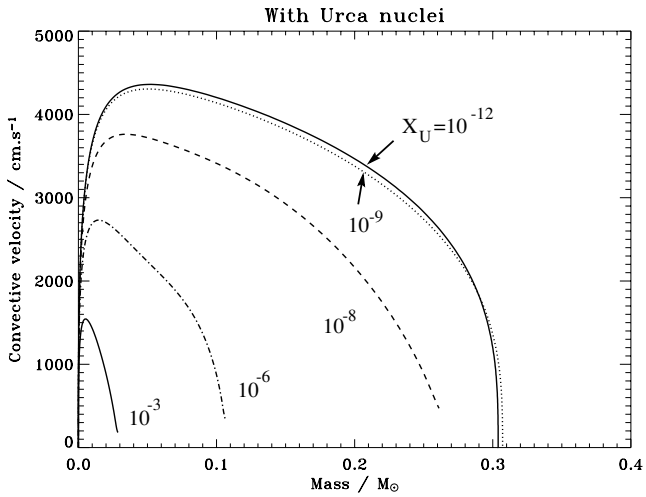

Figure 7. Convective velocity profile of different models of stationary cores with Urca nuclei. The different values for $X_{\rm U}$ (the total mass fraction of Urca nuclei) are indicated next to each curve.

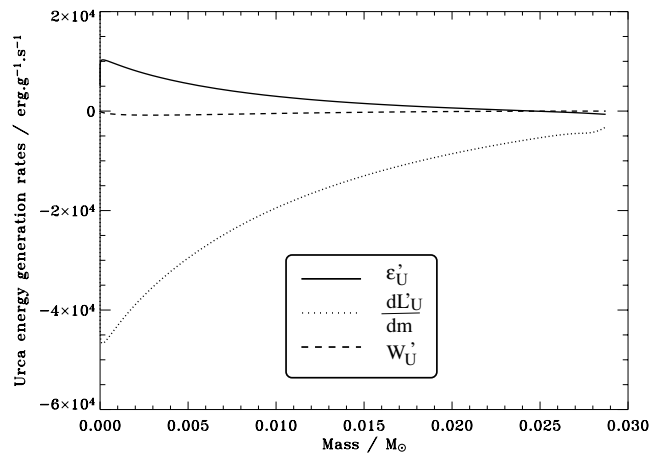

Figure 9. Quantities $\epsilon'_{\rm U}$ (solid), $\partial L'_{\rm U}/\partial m$ (dotted) and $W'_{\rm U}$ (dashed) are plotted for a low central mother fraction case of model $X_{\rm U} = 10^{-3}$ (TSM with work term). The total net energy generation rate is greater than 5×10^4 erg g^{-1} s^{-1}. It is dominated by the energy generation due to carbon burning.

5.3.3 High Urca abundance

For $X_{\rm U} > 10^{-8} \simeq X_{\rm L}$ our shooting method is not able to find a stationary state. When the mother mass fraction at the centre is too high the relative composition difference diverges as the convective velocity tends to zero. When it is too low the convective velocity reaches a minimum and then rises again up to the edge of the white dwarf. Fig. 7 shows the convective velocity profiles in the latter case and demonstrates how drastically the Urca nuclei can affect the velocity profile.

We illustrate this effect by plotting the density difference $\Delta \ln \rho$ in Fig. 8 for a case where $X_{\rm M}$ at the centre is too low. Indeed, the density difference controls the buoyancy force and hence the convective velocity. This figure shows the competing dependence of $\Delta \ln \rho$ on the temperature and on the chemistry (the Urca nuclei). As was previously noted in Section 4.2, the chemical part of the density difference is positive and stabilizes convective motions. In the outer parts of the convective region, the temperature and chemical dependence cancel each other to yield a very small convective velocity. This suggests that the criterion for semiconvection might be fulfilled in the region just above the point where u vanishes.

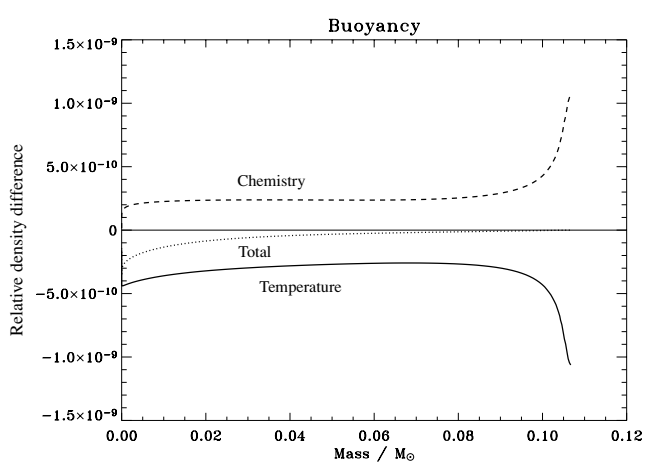

Figure 8. Density difference $\Delta \ln \rho$ through the convective core for a low central mother fraction case. $\Delta \ln \rho = -\delta \Delta \ln T + \boldsymbol{\mu}'' \cdot \Delta \ln \boldsymbol{N}$. It is the sum of a temperature-dependent term and a chemistry-dependent term. We plot $-\delta \Delta \ln T$ (solid line), $\boldsymbol{\mu}'' \cdot \Delta \ln \boldsymbol{N}$ (dashed line) and $\Delta \ln \rho$ (dotted line).

In degenerate matter the mass density is tied to the electron abundance. Hence, all reactions that change the number of electrons have an effect on the buoyancy. The terms $\epsilon_{\rm U}$, $\partial L'_{\rm U}/\partial m$ and $W'_{\rm U}$ seem to be of secondary importance compared to the change in the convective velocity. We plot them in Fig. 9. They are relatively unimportant compared to the energy generation rate due to carbon burning.

6 DISCUSSION

In principle, the formalism we have derived is sufficiently general that it can be used with any model for the exchange (or diffusion) between the two streams. It automatically guarantees conservation of the chemical species and energy and allows time dependence. The formalism also accounts for the interactions between the global contraction or expansion (with velocity v) of the star and the convective (or drifting) velocity u. In fact, the drifting velocities for the upward and downward motion ($v_{\rm u}$ and $v_{\rm d}$) do not even have to be equal.

However, it is only a formalism and requires a model for all horizontal motions/exchanges. In Section 2.5 we suggested a model that is easy to implement and which can be used to compare our new formalism with previously derived theories of convection. However, this has a few limitations.

First, we used a very simple model (where Δu was set equal to zero) for the net mass transfer $\dot{M}$ between the upward and downward moving fluids. This eliminates the main non-local terms involving Δu in equation (48) and hence disables the effects of convective overshooting. To account for overshooting in a self-consistent way requires an a priori physical model for the exchange term $\dot{M}$. However, one can investigate the effects of overshooting within our framework by using a necessarily somewhat ad hoc prescription for the convective velocity, which allows a finite convective speed beyond the formally convective region according to the Ledoux criterion.

Secondly, we chose reversible processes for the exchange of momentum and the mixing of chemical species. Irreversible processes would dissipate part (if not all) of the work done as heat. The cooling

terms $W_{\rm conv}$ and $W'_{\rm U}$ would then have lower (possibly zero) values. For example, the Reynolds numbers in convective regions are so large that there is almost certainly a large amount of turbulence. To account properly for turbulent dissipation in the derivation of our model would require the inclusion of a finite viscosity introduced by the turbulent cascade. This is not easily done. However, one can investigate the possible outcome of dissipation by switching the work terms on and off. In Section 5.3.1 (Fig. 5), we investigated the effect of setting $W_{\rm conv} = 0$ and found differences of up to 20 per cent. Furthermore, setting $W'_{\rm U} = 0$ would most likely only have a small effect because $W'_{\rm U}$ is already small.

6.1 Stationary cores

As a first illustration of our two-stream formalism we computed stationary convective cores. However, the convective core during the carbon flash grows very rapidly. Indeed, the time dependence does matter at least in the energy equation, as was shown in Section 5.1. Our stationary cores may therefore only be very rough approximations of growing convective cores. Moreover, in the present study we used arbitrary abundances for the Urca nuclei. Most of the Urca nuclei are byproducts of carbon burning. To compute their abundances in a self-consistent manner involves quite an extended network, as was shown by Iben (1978a).

Despite these limitations, the results already help to shed some light on the convective Urca process and prove useful in calibrating numerical aspects in the implementation of the method. For example, the terms $\Delta \boldsymbol{R}$, $\Delta\epsilon$ and ΔL, as well as the non-local term in equation (48), are found to be second-order terms in our computations of stationary Urca cores. Furthermore, the chemical composition differences between the two streams are generally small, except possibly at the outer edge of a convective core with a high Urca abundance. This may provide some justification for the use of the one-stream approximation to describe the convective Urca process in a stellar evolution code.

6.2 Future work

The next step is to implement the TSM (or a more simple but appropriate approximation) in a stellar evolution code. Coupled with a suitable nuclear reaction network, this will allow us to follow the evolution of the core in a fully self-consistent manner to the runaway phase in a SN Ia and to determine the physical and chemical conditions in the core at the time of the explosion. This will provide the ignition conditions, the thermodynamic properties and the location of the ignition point for explosion calculations (e.g. Hillebrandt et al. 2004). Moreover, with this tool, we shall be able to systematically address the dependence of the ignition conditions on the overall metallicity, the initial C/O ratio, the white dwarf accretion rate, the initial mass of the white dwarf, etc.

In this context, we note the importance of the neutron excess at the time of the explosion. Timmes, Brown & Truran (2003) have recently emphasized the role of the initial ^{22}Ne abundance, which they argued was determined by the metallicity, introducing a metallicity dependence for SNe Ia. However, the neutron excess itself is affected by the electron captures and emissions in the simmering phase preceding the nuclear runaway. If we take as indicative the Urca abundances derived by Iben (1982, table 4) at the end of his computations, we can compute the number of additional neutrons introduced by the Urca reactions alone on Ne isotopes, $n_{\rm U} = 3Y(^{23}{\rm Ne}) + 5Y(^{25}{\rm Ne}) = 3.6 \times 10^{-3}$. The corresponding number of additional neutrons caused by ^{22}Ne resulting from solar abundances in the white dwarf progenitor is $n_\odot = 2\,Y_\odot\,(^{22}{\rm Ne}) = 2.5 \times 10^{-3}$. Hence, the effect of the Urca isotopes of Ne on the neutron excess can be even larger than that of fossil ^{22}Ne. This provides another illustration for the importance of a proper treatment of the Urca process for answering some of the fundamental, unsolved questions concerning SNe Ia.

7 SUMMARY AND CONCLUSIONS

We have derived a two-stream formalism which carefully addresses the energy and chemical budgets in a convective region. In addition, it allows time dependence and describes the interaction of convection with the general motions of the star. We illustrated this formalism with a simple model and compared the resulting theory of convection to existing theories. We also derived a one-stream limit approximation which will be easy to implement in a stellar evolution code as an extension of classical MLT.

We then applied this formalism to the convective Urca process and derived the entropy equation which has been central to previous discussions of the Urca process, and computed the convective velocity. We showed that the net heating effect of the Urca process strongly depends on the state of mixing of the convective core, for which we provide an estimate. Urca reactions generally tend to reduce the effects of buoyancy. More generally we show that, in degenerate matter, reactions that change the number of electrons have a direct influence on the convective velocity.

As an illustration of our model we computed stationary convective cores. These computations show that, even for a very small Urca fraction, convective velocities are strongly modified compared to the case without Urca nuclei. They also show that convective Urca cores are unlikely to be in a stationary state. Hence, time-dependent computations with a full nuclear reaction network are needed to provide the final answer to the question, what is the influence of the convective Urca process on the ignition conditions in SNe Ia?

ACKNOWLEDGMENTS

PL would like to thank Dr Mike Montgomery for fruitful discussions. We also thank the referee for pertinent remarks. This work was funded through a European Research and Training Network on Type Ia Supernovae (HPRN-CT-20002-00303).

REFERENCES

Barkat Z., Wheeler J. C., 1990, ApJ, 355, 602
Bisnovatyi-Kogan G. S., 2001, MNRAS, 321, 315
Böhm-Vitense E., 1958, Z. Astrophys., 46, 108
Bruenn S., 1973, ApJ, 183, L125
Cannon R. C., 1993, MNRAS, 263, 817
Couch R. G., Arnett W. D., 1975, ApJ, 196, 791
Eggleton P. P., 1971, MNRAS, 151, 351
Eggleton P. P., 1983, MNRAS, 204, 449
Gamow G., Schönberg M., 1941, Phys. Rev., 59, 539
Grossman S. A., Narayan R., Arnett D., 1993, ApJ, 407, 284
Gutiérrez J., García-Berro E., Iben I., Isern J., Labay J., Canal R., 1996, ApJ, 459, 701
Hillebrandt W., Reinecke M., Schmidt W., Roepke F. K., Travaglio C., Niemeyer J. C., 2004, in Warnecke G., ed., Analysis and Numerics of Conservation Laws. Springer, Berlin, in press
Iben I., 1978a, ApJ, 219, 213
Iben I., 1978b, ApJ, 226, 996
Iben I., 1982, ApJ, 253, 248
Kippenhahn R., Weigert A., 1990, Stellar Structure and Evolution. Springer-Verlag, Berlin

Kuhfuss R., 1986, A&A, 160, 116
Mochkovitch R., 1996, A&A, 311, 152
Nomoto K., Sugimoto D., 1977, PASJ, 29, 765
Paczyński B., 1972, Astrophys. Lett., 11, 53
Pols O. R., Tout C. A., Eggleton P. P., Han Z., 1995, MNRAS, 274, 964
Stein J., Barkat Z., Wheeler J. C., 1999, ApJ, 523, 381
Timmes F. X., Brown E. F., Truran J. W., 2003, ApJ, 590, 83
Tsuruta S., Cameron A. G. W., 1970, Ap&SS, 7, 374
Unno W., 1967, PASJ, 19, 140

APPENDIX A: STABILITY ANALYSIS OF CONVECTION

In this appendix we analyse the stationary states of convection when the evolution of the average quantities is very slow. We first investigate the possible available stationary states and then study their linear stability.

In the following we assume that $\Delta \boldsymbol{R} = \Delta\epsilon = \Delta L = \Delta u = 0$ and we neglect v and non-local terms in the difference equations.

A1 Stationary states

The radiative state ($u = 0$) is always a stationary solution of our equations. The other stationary convective states are solutions of the system of equations (55)–(57) when $u = 0$ has already been factored out. We now compute the solutions of this system and examine their existence and possible multiplicity.

A1.1 Cubic equation for the convective velocity

Standard thermodynamics gives

$$\begin{aligned}\frac{\partial e}{\partial r} + p\frac{\partial}{\partial r}\frac{1}{\rho} &= T\frac{\partial s}{\partial r} + \boldsymbol{\mu}\cdot\frac{\partial \boldsymbol{N}}{\partial r}\\ &= Tc_p\left(\frac{\partial \ln T}{\partial r} - \nabla_a\frac{\partial \ln p}{\partial r}\right) + \boldsymbol{\mu}'\cdot\frac{\partial \boldsymbol{N}}{\partial r}\\ &= -\frac{Tc_{\mathrm{P}}}{H_p}(\nabla - \nabla_a) + \boldsymbol{\mu}'\cdot\frac{\partial \boldsymbol{N}}{\partial r},\end{aligned} \tag{A1}$$

where s is the specific entropy, $\nabla_a = (\partial \ln T/\partial \ln P)_{s,N}$, $c_p = (\partial h/\partial T)_{p,N}$, $\boldsymbol{\mu}$ are the chemical potentials and $\boldsymbol{\mu}' = \boldsymbol{\mu} + T(\partial s/\partial \boldsymbol{N})_{T,p}$. We also have

$$\Delta h = c_p\Delta T + \boldsymbol{\mu}'\cdot\Delta\boldsymbol{N}. \tag{A2}$$

Using relation (57), we obtain

$$\frac{\Delta T}{T} = \frac{\lambda}{2H_p}(\nabla - \nabla_a)\left(1 + \beta\frac{\chi}{u\lambda}\right)^{-1} \tag{A3}$$

where $\nabla = \mathrm{d}\ln T/\mathrm{d}\ln p$ and $H_p^{-1} = -(\mathrm{d}\ln p/\mathrm{d}r)$. We now identify the mixing length of MLT with λ, so that equation (A3) becomes the MLT excess temperature equation. The density difference can be written in terms of temperature and abundance differences if we use the thermodynamical relation

$$\Delta\ln\rho = -\delta\Delta\ln T + \boldsymbol{\mu}''\cdot\Delta\ln\boldsymbol{N}, \tag{A4}$$

where $\delta = -(\partial \ln\rho/\partial \ln T)_{p,N}$ and $\boldsymbol{\mu}'' = (\partial \ln\rho/\partial \ln \boldsymbol{N})_{p,T}$. We obtain the convective velocity u by substituting this expression into equation (55) and using relations (57) and (A3)

$$\begin{aligned}&u^3 + u_0u^2 + u_1^2[\delta(-\nabla + \nabla_a) + \boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N]u\\ &\quad + u_0u_1^2\boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N = 0,\end{aligned} \tag{A5}$$

where $u_0 = \beta(\chi/\lambda)$, $u_1 = \sqrt{p/\rho}\lambda/2H_p$, and $\boldsymbol{\nabla}_N = \mathrm{d}\ln\boldsymbol{N}/\mathrm{d}\ln \mathrm{p}$. Note that the dot product $\boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N$ corresponds to the more familiar $\phi\nabla_\mu$ where μ is mean molecular weight, $\phi = (\partial \ln\rho/\partial \ln\mu)_{p,T}$ and $\nabla_\mu = \mathrm{d}\ln\mu/\mathrm{d}\ln p$. Solving this cubic equation for the real positive roots (if they exist) gives the convective velocity, which in turn allows us to compute all the Δ quantities:

$$\Delta\ln T = \frac{\lambda}{2H_p}\frac{u}{u + u_0}(\nabla - \nabla_a) \tag{A6}$$

and

$$\Delta\ln\boldsymbol{N} = \frac{\lambda}{2H_p}\boldsymbol{\nabla}_N. \tag{A7}$$

We now determine the number of solutions of the cubic (A5).

A1.2 Multiplicity of stationary states

This cubic may have three real roots or one real root and two complex conjugate roots according to the sign of its discriminant (positive or negative).

The number of real positive roots relies on the respective signs of $\delta(\nabla - \nabla_a) - \boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N$ (Ledoux criterion) and $\boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N$ (Rayleigh–Taylor criterion). We use the rule of Descartes to obtain the number of solutions.

(i) If $\boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N < 0$, there is one real positive root (Rayleigh–Taylor instability).

(ii) If $\boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N > 0$ and $\delta(\nabla - \nabla_a) - \boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N < 0$, there is no real positive root.

(iii) If $\boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N > 0$ and $\delta(\nabla - \nabla_a) - \boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N > 0$, there is no positive root if the cubic discriminant is positive and there are two positive roots otherwise (convective case).

In the convective case, when $u \gg u_0$, the maximum convective velocity becomes

$$u = u_1\sqrt{\delta(\nabla - \nabla_a) - \boldsymbol{\mu}''\cdot\boldsymbol{\nabla}_N}. \tag{A8}$$

A2 Stability

To analyse the stability of the stationary states we need the time-dependent equations for the convective properties

$$\frac{\lambda}{2}\frac{\partial u}{\partial t} = -u_1^2\frac{2H_p}{\lambda}(-\delta\Delta\ln T + \boldsymbol{\mu}''\cdot\Delta\ln\boldsymbol{N}) - u^2, \tag{A9}$$

$$\frac{\lambda}{2}\frac{\partial\Delta\ln T}{\partial t} = u\frac{\lambda}{2H_p}(\nabla - \nabla_a) - (u + u_0)\Delta\ln T \tag{A10}$$

and

$$\frac{\lambda}{2}\frac{\partial\Delta\ln\boldsymbol{N}}{\partial t} = u\left(\frac{\lambda}{2H_p}\boldsymbol{\nabla}_N - \Delta\ln\boldsymbol{N}\right). \tag{A11}$$

In equation (A10) we assume that the evolution of the mean fluid is slow compared to the convective time-scale so that time derivatives of c_p, $\boldsymbol{\mu}'$ and p can be neglected.

The linear stability of this system at a stationary point is determined by the eigenvalues of the matrix

$$\frac{\lambda}{2}\begin{bmatrix} -2u & \delta u_1^2\dfrac{2H_p}{\lambda} & -u_1^2\dfrac{2H_p}{\lambda}\boldsymbol{\mu}'' \\ \dfrac{\lambda}{2H_p}(\nabla - \nabla_a) - \Delta\ln T & -(u + u_0) & \mathbf{0} \\ \dfrac{\lambda}{2H_p}\boldsymbol{\nabla}_N - \Delta\ln\boldsymbol{N} & \mathbf{0} & -u.\mathbf{I} \end{bmatrix} \tag{A12}$$

where $\mathbf{I}$ is the identity $N \times N$ matrix (N is the total number of species), $\boldsymbol{\nabla}_N$ and $\Delta \ln \boldsymbol{N}$ are vertical N-component vectors, and $\boldsymbol{\mu}''$ is a horizontal N-component vector. The system is stable when the real part of all eigenvalues is negative. It is unstable otherwise.

A2.1 Radiative state ($u = 0$)

In this case the matrix (A12) reads

$$\frac{\lambda}{2}\begin{bmatrix} 0 & \delta u_1^2 \dfrac{2H_p}{\lambda} & -u_1^2 \dfrac{2H_p}{\lambda}\boldsymbol{\mu}'' \\ \dfrac{\lambda}{2H_p}(\nabla - \nabla_a) & -u_0 & \mathbf{0} \\ \dfrac{\lambda}{2H_p}\boldsymbol{\nabla}_N & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{A13}$$

Its characteristic polynomial is $P_N(x) = -(-x)^{N-1} P(x)$ where P is the cubic (A5). Hence, when there exists a convective state (a real positive root of the cubic P) the radiative state must be unstable. If there is no real positive root to P and the discriminant is positive, the stability depends on whether the real part of the complex conjugate roots are negative. The unstable case corresponds to semiconvection, a state that is intrinsically time-dependent. In all the remaining cases (negative discriminant and no real positive root) the radiative state is stable.

A2.2 Convective state ($u > 0$)

In this case, the matrix (A12) reads

$$\frac{\lambda}{2}\begin{bmatrix} -2u & \delta u_1^2 \dfrac{2H_p}{\lambda} & -u_1^2 \dfrac{2H_p}{\lambda}\boldsymbol{\mu}'' \\ -\dfrac{u_0}{u+u_0}\dfrac{\lambda}{2H_p}(\nabla - \nabla_a) & -(u_0 + u) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -u.\mathbf{I} \end{bmatrix}. \tag{A14}$$

When there are two available stationary states with different velocities u the state with the lower velocity is always unstable, and the state with the higher velocity is always stable.

If $u \gg u_0$ the eigenvalues are real negative $\{-2u, -u\}$ and the convective state is stable.

This paper has been typeset from a TEX/LATEX file prepared by the author.