(Semi-)Analytical solutions of 1D partial differential equations

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Outline

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Motivation

Astrophysical gases are a complicated business...

- Rich microphysics
- Radiation transport (cooling, thermal diffusion)
- Chemistry
- Magnetic fields
- High Mach flows
- High resolution features

$\Rightarrow$ Need complicated benchmarks to test codes

The general method

General method

- Consider a set of partial differential equations

$$\mathscr{F}[x, t, y(x,t), \partial_t y, \partial_x y, \partial_{xx} y, ...] = 0$$

- Choose a form for the solution $y(x,t)$
- Plug it in the equations...

And if it doesn't work ?

- Adjust parameters of the solution
- Choose better initial or boundary conditions
- Fudge the physics (change $\mathscr{F}$)

Fourier modes

A Fourier mode

$y = y_0 e^{(s.t+ik.x)}$

- A must for linear equations...

Example

The heat transport equation

$\partial_t y = \partial_{xx} y \Rightarrow s + k^2 = 0$

(dispersion relation $\equiv$ fitting parameters)

- Can work for non-linear equations as well !

Fourier modes

For non-linear PDEs

Examples

- Torsional Alfvén waves for ideal MHD
- Incompressible MRI modes are solutions of the non-linear isothermal ideal MHD Hill system (Note: it needs initially homogeneous conditions).
- For linear cooling and a proper choice of parameters, you get solutions with cooling, resistivity and viscosity as well (Lesaffre & Balbus, 2008) !

JWKB approximation

JWKB expansion

$$y = \exp[st + \frac{1}{\varepsilon} \sum_{n=0}^{n=\infty} \varepsilon^n S_n(x)]$$

- Find constraining ODEs for the S_n at each order in ε.
- Then let $\varepsilon \to 1$.
- The series is sometimes finite or converges very fast.

Example

Sheared waves for the hydrodynamical Hill system (Balbus & Hawley 2006, ApJ)

Non-Linear waves

Non-linear wave form

$$y = Y(s.t - k.x)$$

Injecting this solution can convert the PDE into an ODE for Y.

Examples

- HD sheared waves (Fromang, Papaloizou 2007, A&A)
- Analytical Steady-state shocks (Lesaffre 2006, GAFD)
- Method of characteristics

Self-Similar solutions.

General self-similar form

$y = f(t) + g(t).Y[a(t) + b(t).x]$

- Inject that form in the PDE
- Then request variable separation between t and $X = a(t) + b(t).x$
- That yields one ODE for each $a(t)$, $b(t)$, $f(t)$, $g(t)$ and $Y(X)$

Notes

- Non-linear waves are obtained by this procedure
- Variable separation is more general than dimensional analysis
- Although the time coefficients are often found to be power laws or exponentials

Detailed example

Adimensional equation for convective/radiative transport

$$\partial_t \ln T = \partial_x \mathscr{L}(\partial_x \ln T) \text{ where } \mathscr{L}(\nabla) = \nabla + U.[\nabla - \nabla_{\mathrm{a}}]_+^{3/2}$$

- We first inject the form $\ln T = f(t)F[a(t)x]$ into the equation:

$$\dot{f}\,(F + \frac{f}{\dot{f}}\,\frac{\dot{a}}{a}\,X\,\partial_X F) = a\,\partial_X \mathscr{L}(f\,a\,\partial_X F) \text{ where } X = a\,x$$

- Request variable separation $\Rightarrow$ get time constraints

$$\begin{cases} f\,\dot{f} = \alpha \\ fa = \beta \end{cases} \quad \text{with } \alpha \text{ and } \beta \text{ constants}$$

- and a second order ODE for the shape function $F(X)$:

$$\alpha(F - X\,\partial_X F) = \beta\,\partial_X \mathscr{L}(\beta\,\partial_X F)$$

Notes

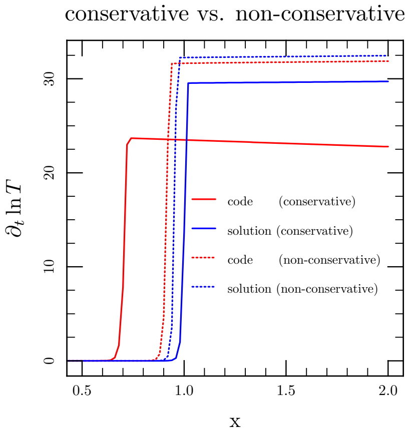
- You get the solution for any function $\mathscr{L}$ (pick up your favourite convection theory)
- $a(t) \propto t^{-2}$ at large times: recover dimensional analysis result for the radiative case
- I believe the present family of solutions has dimension (3+2-2)
- You can find broader families by using all $a(t)$, $b(t)$, $f(t)$, $g(t)$
- You can include nuclear heating provided it is of the form $\varepsilon = a(t)E(X)$

Exercise

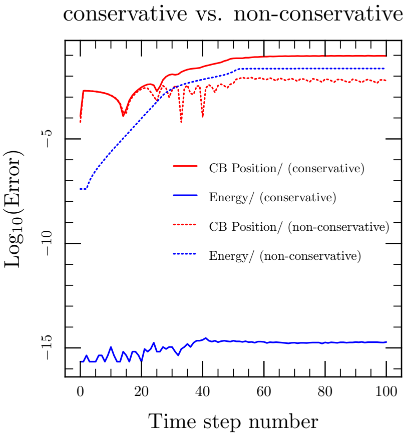
Apply technique to Navier-Stokes equations and recover shock-tube, blast waves, bubbles, exponential solutions (c.f. Zel'dovich & Raizer) with viscosity.

Code Results

Comparing two ways of discretizing the energy equation



(a) Snapshot after 80 time steps



(b) Evolution of errors

Outline

Testing codes

Trust a code's result only when you already know it...

- Check codes, debug them
- Improve their accuracy
- Find their domains of application
- Probe their convergence properties

Example

Lesaffre & Balbus (2007) probe the dissipation properties of ZEUS thanks to Alfvén waves with viscosity and resistivity. They measure a scaling law for the total dissipation:
$\eta_{\rm N} + \nu_{\rm N} = 0.76(\frac{k}{2\pi})^{1.6}\Delta x^2 \beta^{-1/2} + 1.08 \Delta x\, C\, \beta^{-1}$
where k is the wave number, Δx is the space resolution, C is the Courant number and β is the plasma-β parameter

Adress new physics

- Basis for linear analysis
  - parasitic modes
  - frozen modes
- Get physical behaviour for a wide range of parameters
- Understand continuity / discontinuity of solutions

Example

$\partial_{xx} T$ and $\partial_x N$ are discontinuous at convective/radiative boundaries (on self-similar solutions of the heat and chemical transport problem)

Build new numerical algorithms

- First find a large family of solutions to your problem
- Carefully assess the dimension of that class (beware of degeneracy, numerical and mathematical)
- The analytical solutions space needs to be dense in the solutions space
- Carefully design a way of picking up only one analytical solution (given initial and boundary conditions)
- Assess the stability of the process

Example: Godunov schemes

- Assume initial conditions=smooth+Riemann problem.
- Evolve thanks to self-similar solution.
- Reconstruct the profile to get smooth+Riemann

Summary

- It is possible to find large classes of (semi-)analytical solutions, even for complicated problems.
- They can be really useful...
  - to probe codes
  - to understand physics
  - to build new algorithms