Exact channel solutions with viscosity, resistivity and cooling

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Outline

1) Shearing sheet equations

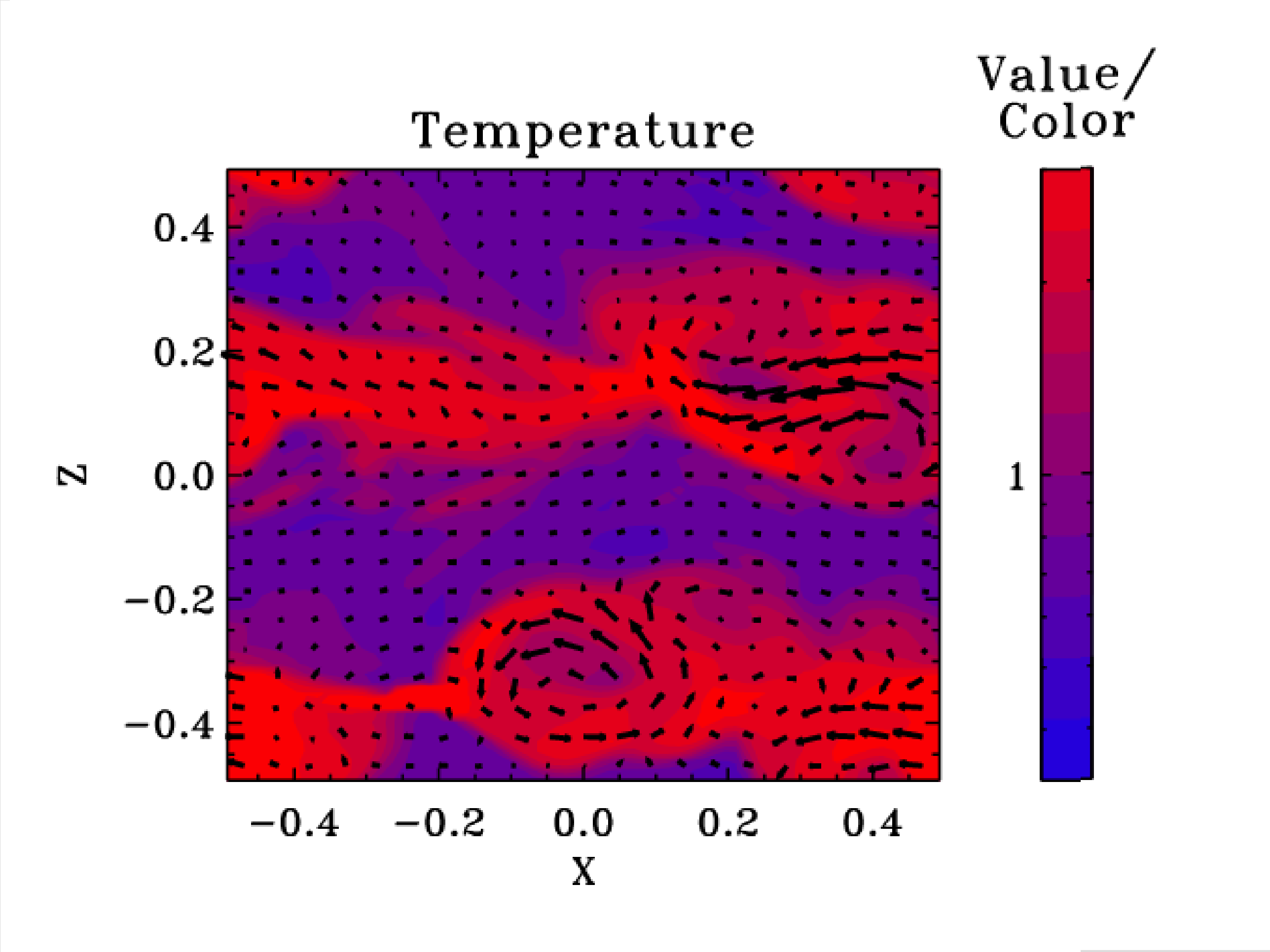
2) Linear Solutions

3) Code design & Benchmarks

4) Numerical viscosity and resistivity

Motivation: more microphysics in codes

Example: study of **active** and **dead** phase separation in a local simulation with temperature-dependent resistivity.



(1) SHEARING SHEET Equations

Continuity

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\boldsymbol{v} = 0 \qquad (1)$$

Euler

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + 2\Omega\hat{\boldsymbol{z}} \times \boldsymbol{v} - \boldsymbol{\nabla}(2A\Omega x^2) + \frac{1}{\rho}\boldsymbol{\nabla}\cdot(P + \frac{B^2}{2}) - \frac{1}{\rho}\boldsymbol{B}\cdot\boldsymbol{\nabla}\boldsymbol{B} = \frac{1}{\rho}\boldsymbol{\nabla}\cdot[\rho\nu_V(\boldsymbol{D} - \frac{1}{3}\boldsymbol{\nabla}\cdot\boldsymbol{v})] \qquad (2)$$

Induction

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{B}\boldsymbol{\nabla}\cdot\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{\nabla}\times(\eta_B\boldsymbol{J}) \qquad (3)$$

Energy

$$\frac{P}{\gamma - 1}\frac{\mathrm{D}\ln(P\rho^{-\gamma})}{\mathrm{D}t} = \eta_B J^2 + \rho\nu_V\boldsymbol{Q} : \boldsymbol{\nabla}\boldsymbol{v} + \Lambda \qquad (4)$$

(2) Trial Solution : steady flow + vertical Fourier mode

$$\boldsymbol{v} = 2Ax\hat{\boldsymbol{y}} + \delta\boldsymbol{u}\ e^{st+ikz} \quad \text{with } \delta\boldsymbol{u} \perp \hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{b}\ e^{st+ikz} \quad \text{with } \delta\boldsymbol{b} \perp \hat{\boldsymbol{z}}$$

Continuity

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\rho\boldsymbol{v} = 0 \tag{1}$$

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Incompressible solution

✓ $$\boldsymbol{v} = 2Ax\hat{\boldsymbol{y}} + \delta\boldsymbol{u}\ e^{st+ikz} \quad \text{with } \delta\boldsymbol{u} \perp \hat{\boldsymbol{z}}$$

$$\boldsymbol{B} = B_0\hat{\boldsymbol{z}} + \delta\boldsymbol{b}\ e^{st+ikz} \quad \text{with } \delta\boldsymbol{b} \perp \hat{\boldsymbol{z}}$$

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Vertical Magnetic field

$$\boldsymbol{v} = 2Ax\hat{\boldsymbol{y}} + \delta\boldsymbol{u}\; e^{st+ikz} \quad \text{with } \delta\boldsymbol{u} \perp \hat{\boldsymbol{z}}$$

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“Linear” Cooling

✓ $$\Lambda = \Gamma - \alpha P$$

(Equivalent to the leading Taylor expansion w.r.t. temperature of the net heating/cooling function)

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Total pressure gradient

Last remaining non linear term

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Euler

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Total pressure evolution

Spatial constant, exponential in time

Need to be zero

$$\frac{\partial P_{\rm tot}}{\partial t} + \alpha P_{\rm tot} = \left[\Re[s] + (\gamma - 1)(\eta - \frac{\alpha}{2})\right]\frac{|\delta b|^2}{2} + (\gamma - 1)\nu\rho\frac{|\delta u|^2}{2} + \Re[a\exp(2st + 2ikz)]$$

with

$$a = \frac{\gamma - 1}{2}\frac{Q}{(s+\eta)^2}\delta {b_x}^2$$

where Q is the following polynomial in s:

$$Q(s) = (\frac{1}{\gamma - 1}s - \eta - \frac{\alpha}{2})[(s+\eta)^2 + \frac{1}{4\Omega^2}(s^2 + \eta s + 4A\Omega + w^2)^2]$$

$$+\frac{\nu}{w^2}(s+\eta)^2[(s+\eta)^2 + (-2A + \frac{1}{2\Omega}((s+\nu)(s+\eta) + 4A\Omega + w^2))^2].$$

=> s has to be a root of Q and the dispersion relation of the system at the same time

=> we get 1 constraint on the parameters ν, η, A, Ω, ω and α

Solutions

- Recover ***torsional Alfvén waves*** in absence of shear
- Generally are standing waves, possibly growing, ie: ***Channel solutions***

- Possible extensions:
  - thermal conduction
  - non zero radial wave numbers

(3) Code design & Benchmarks Start with ZEUS3D

- **Internal energy** equation (Stone & Norman 1992)

$$\frac{\partial e}{\partial t} + \boldsymbol{\nabla}\cdot(e\boldsymbol{v}) + p\boldsymbol{\nabla}\cdot\boldsymbol{v} = \eta(T)J^2 - \boldsymbol{Q}:\boldsymbol{\nabla}\boldsymbol{v} - C(T)$$

- **Total energy** equation (Turner et al. 2003; Hirose et al. 2006)

$$\frac{\partial \mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} = -C(T)$$

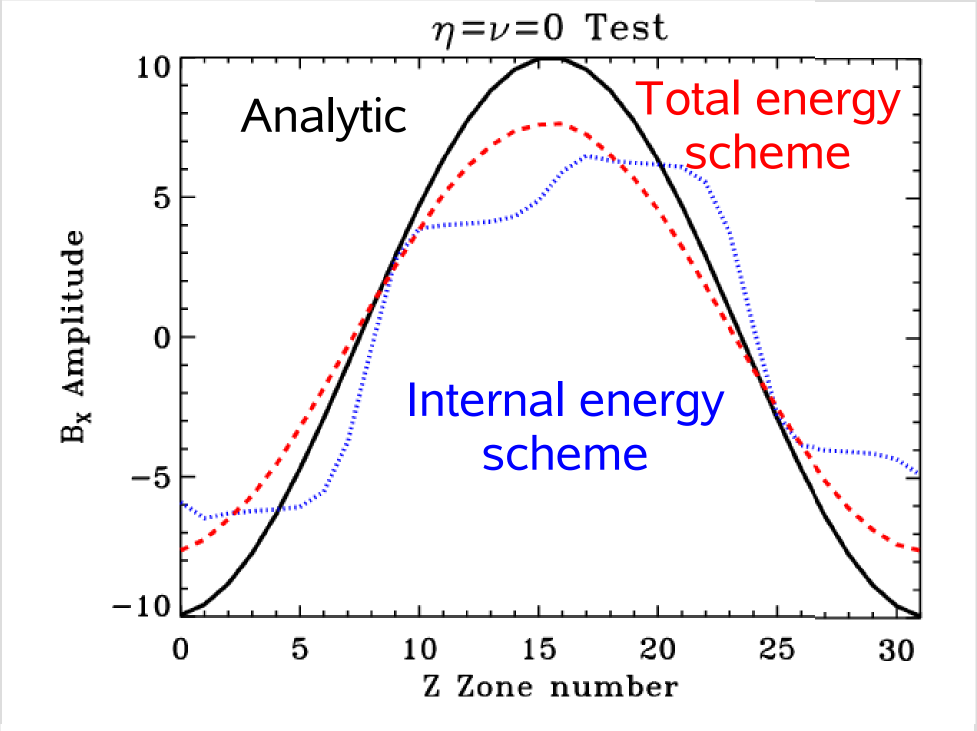
with

$$\mathcal{E} = e + \frac{1}{2}\rho v^2 + \frac{1}{2}b^2 + \rho\Phi$$

and

$$\boldsymbol{\mathcal{F}} = \boldsymbol{v}(e + p + \frac{1}{2}\rho v^2 + \rho\Phi) + \boldsymbol{b}\times\boldsymbol{v}\times\boldsymbol{b} + \eta(T)\boldsymbol{J}\times\boldsymbol{b} + \boldsymbol{Q}\circ\boldsymbol{v}$$

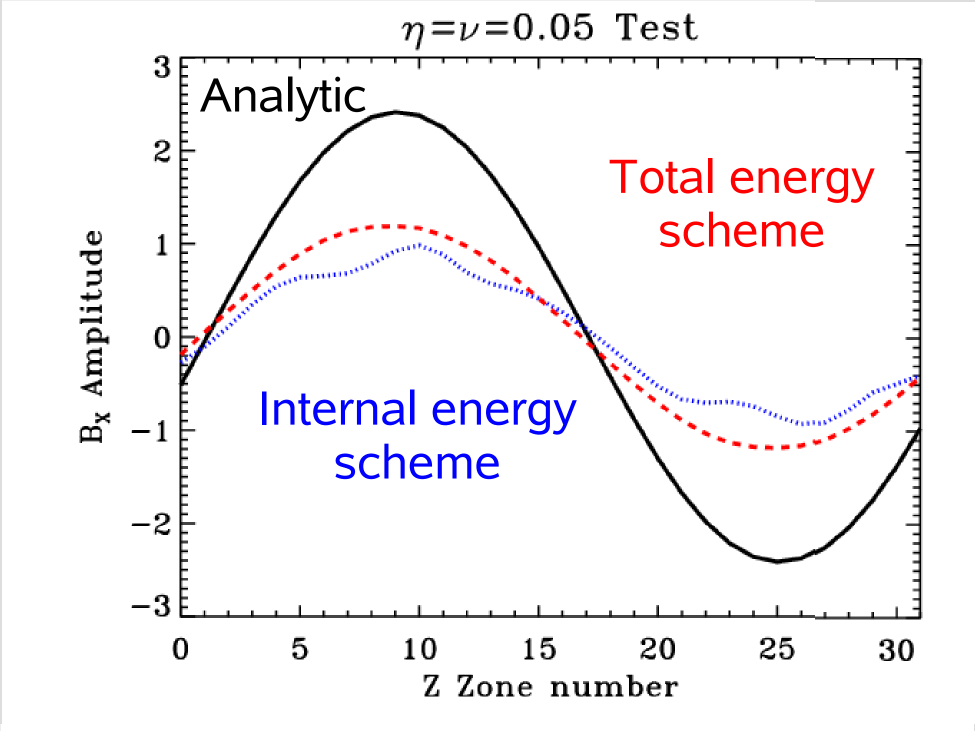
Benchmark: Torsional Alfvén waves



A torsional Alfvén wave after 3 oscillations (see Turner et al. 2003)

- The internal energy scheme **distorts** the wave

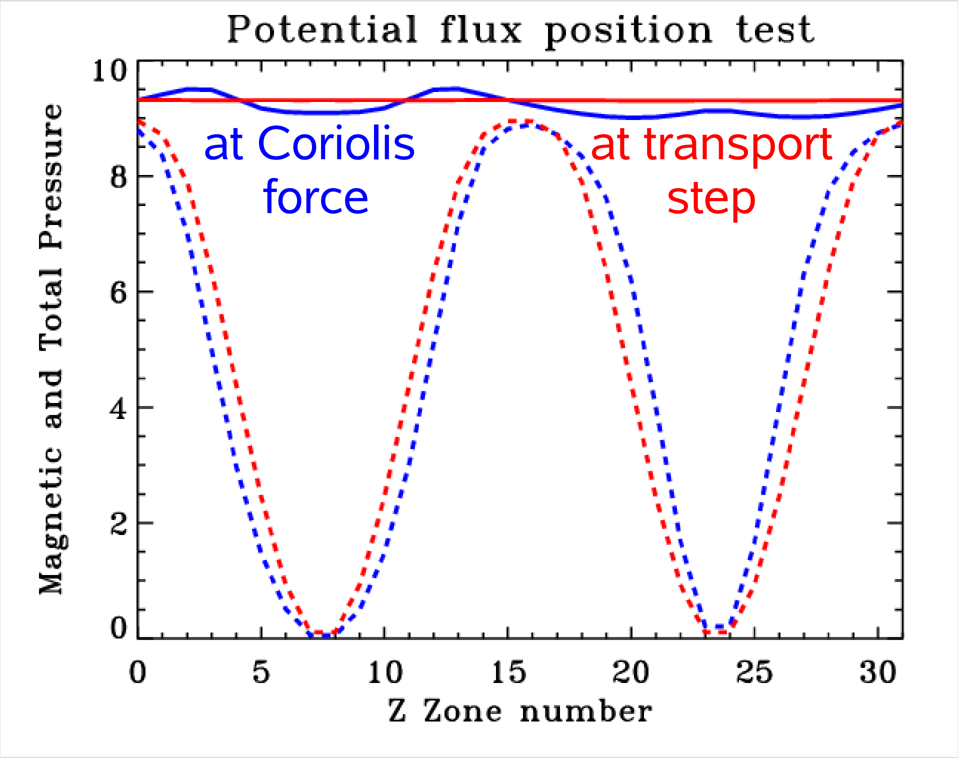
Benchmark: with resistivity and viscosity



A torsional Alfvén wave after 7.2 periods

- Adding some physical viscosity and resistivity improves the internal energy scheme.

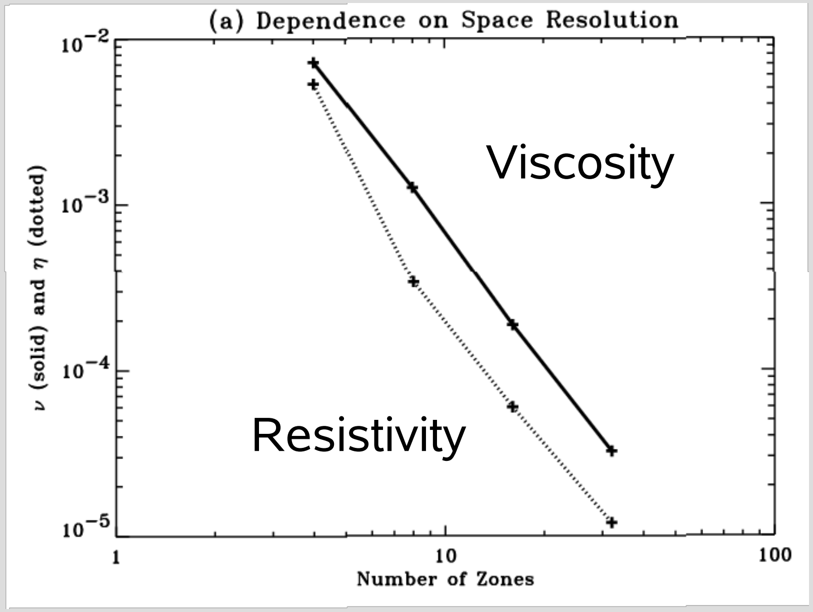
Benchmark: MRI modes with resistivity (and cooling)

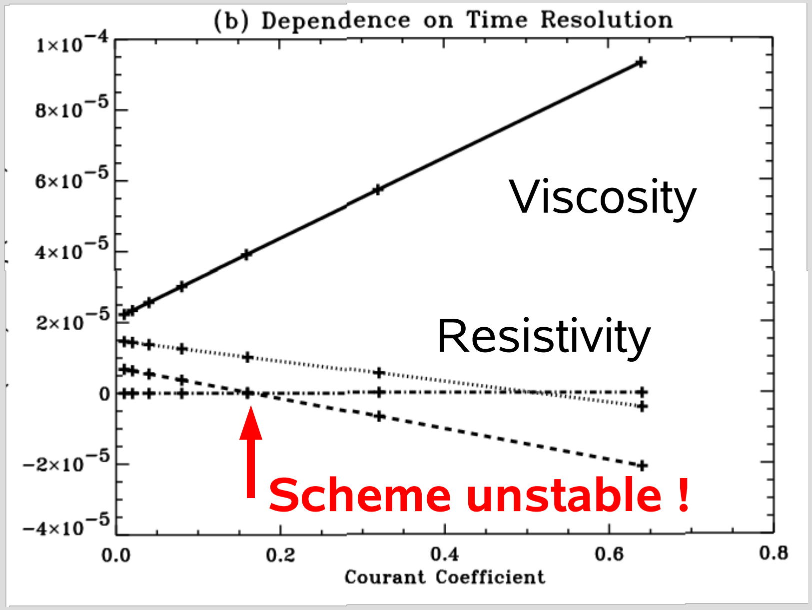


MRI standing mode after 8 orbits

- **Resistivity** is **adjusted** to satisfy total pressure homogeneity
- The potential energy flux should better be computed *along the transport step* rather than associated with the Coriolis momentum source.

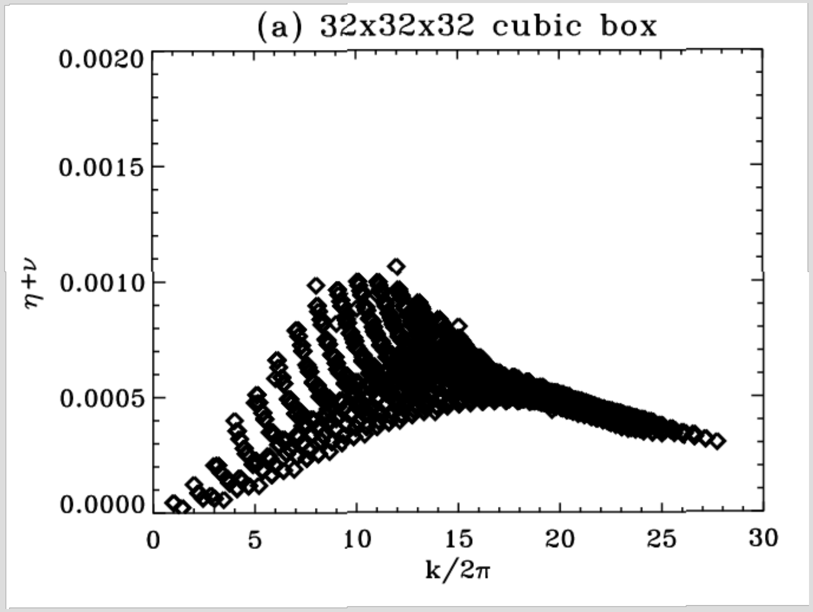
(4) Measuring η and ν

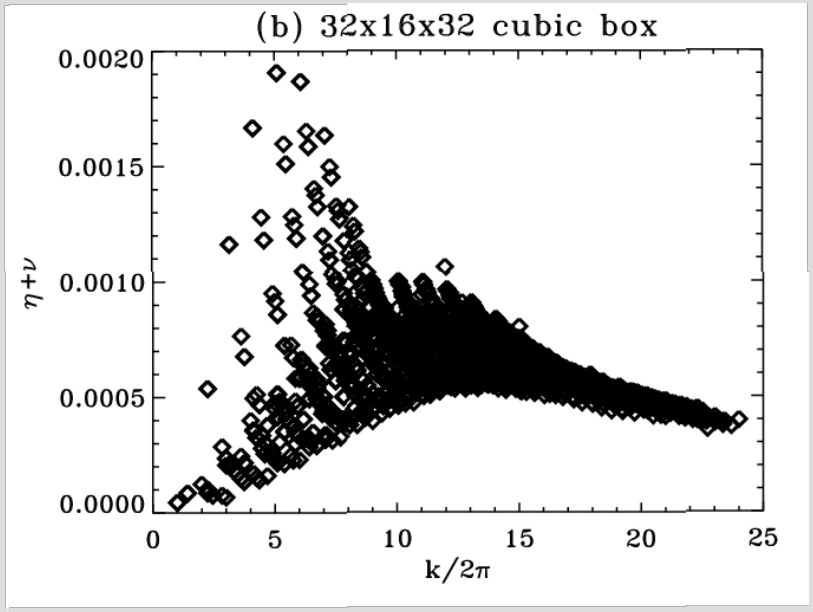




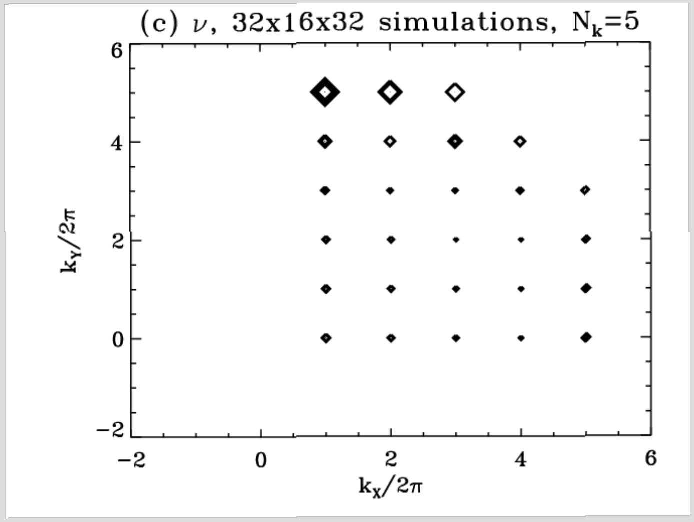
- η+ν can be measured from the **decay**
- η−ν can be measured from the **phase lag** between velocity and magnetic field

Anistropy of numerical dissipation





- Dissipation depends upon **leading wave number**
- Elongated pixels **increase anisotropy**
- They also **increase** the total **dissipation**: not worth the resolution ?
- Dependence on k linked to ***bottleneck effect*** ?



Conclusions

- **New benchmarks**
- **ZEUS3D+...**
- **Thermal stability** of accretion disks
- **Next step: RAMSES-MHD with shearing box and resistivity**
- ***Parasitic instabilities*** with more physics

Why an elongated box ?

- Parasitic instability (Goodman & Xu, 1998) **peaks at ~ a third** of the MRI most unstable mode.